THE JOURNAL
OF THE
INDIAN MATHEMATICAL SOCIETY

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NEW SERIES

Vol. VIII

1944

The Indian Mathematical Society acknowledges with thanks the receipt of financial help from the Rockfeller Foundation in bringing out this volume of the journal.

MADRAS :

PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE.

Annual Subscription : Rs. 6

CONTENTS

	Pages.
B. N. Bose: On some transformations in generalized Hyper-geometric Series	119—128
S. C. Chakrabarti: Some further Algebraic identities..	115—119
K. Chandrasekharan: On Strum-Lionville Series ..	109—114
Ching-Syur Siez: A General expression for Euler's ϕ-Function	91—94
A. C. Choudhury: On 2-Webs of Curves in R_n ..	36—44
S. Chowla and A. M. Mian: The Differential equations satisfied by Certain Functions ..	27—28
V. Ganapathy Iyer: On Singular functions ..	94—108
Harishanker: Some infinite Integrals involving Cylinder Functions	57—60
F. W. Levi: Notes on Group-theory I, II ..	1—9
————: ,, III ..	45—56
————: ,, IV—VI ..	78—91
A. M. Mian and S. Chowla: The Differential equations satisfied by Certain Functions ..	27—28
S. Minakshisundaram: On the Differentiated Series of Eigenfunctions	75—78
S. S. Pillai: On $a^x - b^y = b^y \pm a^x$	10—13
————: On the smallest primitive root of Prime ..	14—17
————: On Waring's Problem with powers of Primes	18—20
————: Highly Composite Numbers of the tth Order	61—74
Shing-Shern Chern: On a Theorem of Algebra and its Geometrical application	29—36
S. Sinha, (Miss): Infinite integrals involving Bessel Functions of imaginary Argument ..	21—26

NOTES ON GROUP THEORY I, II.

BY

F. W. LEVI, *Calcutta*.

[Received 28 February 1944].

I. THE MAPPINGS $a \to a^m$

Let $\{m\}$ be the mapping by which every element a of a group G is represented by its mth power a^m, then $\{0\}$ and $\{1\}$ are endomorphisms for every group, whereas $\{-1\}$ and $\{2\}$ are endomorphisms if and only if G is an Abelian group. If G is a finite group of order n, then $\{kn\}$ and $\{kn+1\}$ are endomorphisms for every integral number k. All these statements are special cases of a single theorem which will be proved in this note and from which several consequences will be deduced.

1. THEOREM. *To every group G there belongs an integral number m_0 with the following properties:*

1. *The numbers m for which $\{m\}$ is an endomorphism form full classes of residues mod m_0 (endomorphic classes).*

2. *If $\{m\}$ is an endomorphism, then is $\{1-m\}$ and $m(1-m) \equiv 0 \pmod{m_0}$; moreover the groups on which G is mapped by the endomorphic class $\equiv m \pmod{m_0}$ are commutative with those on which G is mapped by the class $\equiv (1-m)$.*

3. *$\{m\}$ maps G endomorphically on an Abelian subgroup G_m if and only if $m \equiv 0 \pmod{m_0}$; in this case G_m lies in the centre of G.*

4. *If $\{m\}$ and $\{m+1\}$ are endomorphisms, then $m \equiv 0$ $\pmod{m_0}$.*

5. *If $\{m_1\}$ and $\{m_2\}$ are endomorphisms, then $\{m_1 m_2\}$ is an endomorphism.*

PROOF. If the suppositions (5) hold, then $\{m_1 m_2\}$ is the product of two endomorphisms and is therefore an endomorphism. In general $\{m\}$ is an endomorphism if and only if

$$(ab)^m = a^m b^m \tag{1}$$

for every pair of elements a and b of G. Now $(ab)^m = a(ba)^{m-1}b$ for every integral value of m (also for negative m). Hence (1) is equivalent with $(ba)^{m-1} = a^{m-1}b^{m-1}$ and therefore with

$$(ba)^{1-m} = b^{1-m}a^{1-m}. \tag{1'}$$

Hence if $\{m\}$ is an endomorphism, so is $\{1-m\}$.

If so, $(a, b)^m = (aba^{-1}b^{-1})^m = a^m b^m a^{-m} b^{-m} = (a^m, b^m)$ $= (aba^{-1})^m b^{-m} = ab^m a^{-1} b^{-m}$; hence

$$(a, b)^m = (a^m, b^m) = (a, b^m). \tag{2}$$

From $(a^m, b^m)(b^m, a) = 1$ follows $(a^{1-m}b^m) = 1$. (3)

Hence the elements a^{1-m} on which G is mapped by the endomorphism $\{1-m\}$ are commutative with the elements b^m on which G is mapped by $\{m\}$. Therefore the elements $a^{m(1-m)}$ on which G is mapped by $\{m(1-m)\}$ are commutative one with another; hence they form an Abelian group.

As $\{0\}$ maps G on 1, there are always integers m such that $\{m\}$ is an endomorphism mapping G on an Abelian group G_m. In this case $(a^m, b^m) = 1$, but from (2) it follows that G_m lies in the centre. Let moreover $\{m'\}$ map G on an Abelian group, then a^m, a^{-m}, $a^{m'}$, $a^{-m'}$ all lie in the centre. Now $(ab)^{m-m'} = (ab)^m(ab)^{-m'} = a^m b^m b^{-m'} a^{-m'}$ $= a^{m-m'}b^{m-m'}$ lies in the centre. Hence $\{m-m'\}$ is an endomorphism mapping G into the centre. Hence the integers m for which $\{m\}$ is an endomorphism mapping G on an Abelian group, form a modul of integral

numbers, i.e. they are the multiples of a particular number m_0. Let now $\{m'\}$ be an endomorphism, then $(ab)^{km_0+m'} = (ab)^{km_0}(ab)^{m'} = a^{km_0}b^{km_0}a^{m'}b^{m'}$ and since the first two factors lie in the centre, $(ab)^{km_0+m'} = a^{km_0+m'}b^{km_0+m'}$. Hence the numbers m for which $\{m\}$ is an endomorphism form full classes of residues (mod m_0). Hence the propositions (1) and (3) hold. Of (2), all but the last statement have been proved already. If $\{m\}$ is an endomorphism, then a^{1-m} and b^m are commutative for every a and every b as has been shown by (3); moreover a^{rm_0} and b^{sm_0} lie in the centre. Hence a^{1-m} and b^m are commutative. This proves (2). To prove (4), suppose that $\{m\}$ and $\{m+1\}$ are endomorphisms, then $\{1-(m+1)\} = \{-m\}$ is an endomorphism. Hence $(ab)^{-m} = a^{-m}b^{-m}$; on the other hand, the left side is equal to $(b^{-1}a^{-1})^m = b^{-m}a^{-m}$, $\{m\}$ being an endomorphism. Hence a^m and b^m are commutative and therefore m_0 is a factor of m. This completes the proof.

2. The integral number m_0 is determined up to a factor ± 1 only; it will therefore be supposed without loss of generality that m_0 is non-negative. Using this notation and the earlier ones, we shall now draw some consequences from the theorem and the definitions.

As $(a, b)^{m_0} = a^{m_0}b^{m_0}a^{-m_0}b^{-m_0}$, the commutator group is mapped on 1 by $\{m_0\}$. Hence the order of every element of $C(G)$ must be a factor of m_0. If these orders have no upper limit or if $C(G)$ contains an element of infinite order, then $m_0 = 0$. Hence if G is a free group with more than one generator or the free product of two groups which are both $\neq 1$, then $m_0 = 0$. The lowest common multiple of the order of the elements of the commutatorgroup is not necessarily equal to m_0 as shown by the example of the quaternion-group where $(a, b)^2 = 1$ whereas $m_0 = 4$. If m' is the *l. c. m.* of the orders of the ele-

ments of G and m'' is the *l. c. m.* of the orders of the elements of $C(G)$, then m_0 is a factor of m' and divisible by m''; if in particular $\{m''\}$ is an endomorphism, then $m_0 = m''$. The set of the elements into which G is mapped by any $\{n\}$ is invariant for endomorphisms. Hence if $\{m\}$ is an endomorphism, it maps G on a fully invariant subgroup.

G is Abelian if and only if $m_0 = 1$. The well-known properties of Abelian groups quoted in the introduction can be subsumed under the theorem as follows. From (2), it follows that $\{2\}$ is an endomorphism if and only if $\{-1\}$ is; if so, then [by (5)] $\{(-1)^2\} = \{1\}$ is an automorphism and [by (4)] $m_0 = 1$.

If m_0 is a prime number p, then G is non-Abelian and therefore p is odd. There is no number $1 < m < p$ such that $m(1-m) \equiv 0 (\text{mod } p)$. Hence 0 and 1 are the only endomorphic classes mod p. Similarly when $m_0 = 2p$ and p is prime (2 included). $m(1-m) \equiv 0 \pmod{2p}$ admits only the solutions

$$m \equiv 0, 1, p, p+1.$$
$$1 - m \equiv 1, 0, p+1, p.$$

But from (4) it follows that if p and $p+1$ are endomorphisms, $m_0 = p$, contrary to the supposition.

3. Let $\{3\}$ be an endomorphism of an non-Abelian group G. Then [by 2], m_0 must be factor of 6 and as 1, 2 and 6 are ruled out by the above considerations $m_0 = 3$. The cubes lie all in the centre and therefore $(a, b)^3 = (a^3, b^3) = 1$. Moreover $(ab)^2 = b^2a^2$ and therefore $(abc)^2 = c^2b^2a^2$.

Now $(a, b) = (ab)^2(b^{-1}a^{-1}b)^2b^{-2}$. Hence $(a, b) = (b^{-1}. b^{-1}a^{-1}b. ab)^2 = [b^{-2}(a^{-1}, b)b^2]^2$. Applying the formula to (a^{-1}, b), we obtain $(a, b) = b^{-4}(a, b)^4b^4$.

As $(a, b)^3 = 1$ and b^3 lies in the centre, $(a, b) = b^{-1}(a, b)b$. This means that (a, b) is commutative with b

for every (a, b). The group G is therefore an L-group[1]. In these groups the commutator-groups are Abelian; the second commutators lie in the centre and their order is 3 or 1. Moreover $(a, b)^t = (a^t, b) = (a, b^t)$. Suppose now that G is a non-commutative L-group in which $(a, b)^3 = 1$ holds for every commutator; it will be shown that $m_0 = 3$. It suffices to show that $\{-2\}$ is an endomorphism. Now for every pair of elements a, b there is

$$(ab)^2 a^{-2} b^{-2} = b(b^{-1}, a)(a^2, b) b^{-1} = b(a, b)^3 b^{-1} = 1.$$

Hence $a^{-2}b^{-2} = (ab)^{-2}$ as proposed. The groups for which $\{3\}$ is an endomorphism can therefore easily be constructed by introducing the condition $(a, b)^3 = 1$ into the representation of the L-groups. Consider e. g. groups which are generated by not more than 3 generators (these groups show already all the essential features). Every L-group of that kind is homomorphic to a group which can be represented by 7 parameters

$$[p, q, r | P, Q, R | e], \tag{4}$$

where the parameter e runs over the classes mod 3 whereas the other parameters run over all the integral numbers. An easy calculation shows that the condition $(a, b)^3 = 1$ for every pair a, b implies that $[0, 0, 0 | 3a, 3b, 3c | 0] = [0, 0, 0 | 0, 0, 0 | 0]$ and this again implies that P, Q, R have to be taken (mod 3). Thus we obtain the non-Abelian groups where $\{3\}$ is an endomorphism and generated by not more than 3 generators, in the form (4) with the law of multiplication

$$[p_1, q_1, r_1 | P_1, Q_1, R_1 | e_1] [p_2, q_2, r_2 | P_2, Q_2, R_2 | e_2] = [p, q, r, | P, Q, R | e]$$

$$p = p_1 + p_2, \; P \equiv P_1 + P_2 - q_2 r_1 \pmod 3$$

$$q = q_1 + q_2, \; Q \equiv Q_1 + Q_2 + p_2 r_1 \pmod 3$$

1. F.W. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, *Jour. Ind. Math. Soc.* (2) 6, (1942), 87.

$$r = r_1 + r_2, \quad R \equiv R_1 + R_2 - p_2 q_1 \pmod 3$$

$$e \equiv e_1 + e_2 + P_1 p_2 + Q_1 q_2 + R_1 r_2 + p_2 q_2 r_1 - p_2 q_1 r_1 - p_2 q_1 r_2 \pmod 3$$

or as groups homomorphic to it.

4. The factors of a direct product remain invariant for a mapping $\{m\}$; it is therefore an endomorphism for $G \times H$ if and only if it is an endomorphism for G and H. Denoting by $E(G)$ the set of numbers m for which $\{m\}$ is endomorphic, we obtain $E(G \times H) = E(G) \cap E(H)$. The number m_0 for $G \times H$ is the *l. c. m.* of the corresponding numbers for G and H. If $\{m\}$ is an endomorphism of G, it is also an endomorphism of all the subgroups of G and of all the groups homomorphic to G, i.e. of all the factorgroups G/N. Thus $E(G) \subseteq E(S)$, $E(G) \subseteq E(G/N)$ for every subgroup S and every normal subgroup N of G. Similarly the number m_0 for G is divisible by the corresponding numbers for S and G/N.

Let u be any odd number > 1, then one can obtain a group with $m_0 = u$ in the following way. These special groups are generated by elements of infinite order; the commutator-group is cyclic of order u and lies in the centre.

Denote elements by $[p, q, r]$ where p and q run over all integral numbers and r is taken (mod u). Define the multiplication by

$$[p, q, r]\,[p', q', r'] = [p+p', q+q', r+r'-p'q].$$

Then $$[p, q, r]^n = [np, nq, nr - \binom{n}{2} pq],$$ therefore $[p, q, r]^u = [up, up, 0]$. $[1, 0, 0]\,[0, 1, 0] = [0, 0, 1]$ and

$$[0, 0, 1]^n = [0, 0, n] \neq [0, 0, 0] \text{ for } 0 < n < u.$$

Thus $\{u\}$ is an endomorphism $[p, q, r] \to [up, uq, 0]$ mapping the group into its centre, and u is the smallest positive number mapping the commutator $[0, 0, 1]$ on the unity-element. Hence $m_0 = u$.

5. If G is any group and m an integral number, then the elements g for which g^m lies in $C(G)$ form a normal subgroup $C(G, m)$, say. If $\{m\}$ maps a and b on the same element, $(ab^{-1})^m = (ab^{-1})^m b^m a^{-m}$ belongs to $C(G)$. Hence $a \equiv b \left(\text{mod } C(G, m)\right)$. The various resolutions of G into mutually exclusive subsets form a partially ordered set Σ of which those generated by endomorphisms form one subset and those generated by mappings $\{m\}$ form another. The meet of those two sets consists of the elements of Σ generated by endomorphic mappings $\{m\}$. Though $\{m\}$ and $\{-m\}$ are in general different mappings, they generate the same element of Σ. $\{km\}$ generates a smaller element than $\{m\}$ and $\{m\}$ a smaller one than $G \to G/C(G, m)$.

II. Mappings $a \to (b, a)$.

Let G be a group and A a subgroup of G. Denote by $\{b\}$ the mapping $a \to (b, a)$ of the elements a of A, and by $\overline{\{b\}}$ the mapping $a \to (a, b)$; let B_A be the set of those elements b of G for which $\{b\}$ is a homomorphism of A. The present note deals with the relations between G, A and B_A and the properties of the groups on which A is mapped by $\{b\}$. From the identity $(b, a_1a_2) = (b, a_1)(b, a_2)\left((a_2, b), a_1\right)$, follows immediately

THEOREM 1. *$\{b\}$ is a homomorphism if and only if*

$$\left((a_1, b), a_2\right) = 1 \tag{1}$$

for every pair a_1, a_2 of elements of A.

THEOREM 2. *If $\{b\}$ is a homomorphism, then is $\overline{\{b^{-1}\}}$.*

PROOF. $(b^{-1}, a_1a_2) = b^{-1}a_1(a_2, b)ba_1^{-1}$. By (1), a_1 is commutative with (a_2, b). Thus

$(b^{-1}, a_1a_2) = b^{-1}(a_2, b)a_1ba_1^{-1} = (b^{-1}, a_2)(b^{-1}, a_1)$. Hence

$$(a_1a_2, b^{-1}) = (a_1, b^{-1})(a_2, b^{-1}). \qquad (2)$$

THEOREM 3. *If $\{b\}$ and $\{b^{-1}\}$ are homomorphisms, then both map A on Abelian groups.*

PROOF. It suffices to show that $\{b^{-1}\}$ maps A on an Abelian group i.e. that (b^{-1}, a_1) $(b^{-1}, a_2) = (b^{-1}, a_2)$ (b^{-1}, a_1) for every pair a_1, a_2 of elements of A. Now the left side is equal to (b^{-1}, a_1a_2) because $\{b^{-1}\}$ is a homomorphism and the right side is equal to the same commutator since $\{b\}$ and therefore $\{b^{-1}\}$ is a homomorphism.

THEOREM 4. *If A is a normal subgroup of G, the elements of B_A map A on Abelian groups with mutually commutative elements.*

PROOF. As A is a normal subgroup of G, the elements (b, a) belong to A; but [by (1)] each (b, a) is commutative with every element of A and therefore with every other (b, a).

THEOREM 5. *$B'_A = B_A \cap A$ is a group and B_A consists of full right hand cosets of B'_A.*

PROOF. Let b' be an element of B'_A and b be an element of B_A. It will be proved that $b'b$ belongs to B_A. Now $(b')^{\pm 1}$ and $a^{\pm 1}$ are commutative with (b, a). Hence

$$(b'b, a) = b'(b, a)\, ab'^{-1}a^{-1} = (b', a)(b, a) \quad (3)$$

for every element a of A. Put $a = a_1a_2$, then $(b'b, a_1a_2)$ $= (b', a_1)(b, a_2)(b, a_1)(b, a_2)$. As (b', a_2) belongs to A it is commutative with (b, a_1). Hence by applying (3) to $a = a_1$ and $a = a_2$ we obtain $(b'b, a_1a_2) = (b'b, a_1)\,(b'b, a_2)$. Therefore $b'b$ belongs to B_A. If in particular b belongs to B'_A, then $b'b$ lies in $B_A \cap A = B'_A$. To show that b^{-1} belongs to B'_A, apply (2) putting b' for b and note that the terms on the right side are commutative; thus

$$(b'^{-1}, a_1a_2) = (b'^{-1}, a_1)(b'^{-1}, a_2).$$

Hence the theorem.

Put in particular $A = G$, then $B_G = B'_G$ is a group. Every endomorphism of G maps B_G into itself: hence B_G is fully invariant. From (1) it follows that $B_G = G$ if and only if $\big((a, b), c\big) = 1$ for every triplet a, b, c of elements of G. Hence we have

THEOREM 6. *$B_G = G$ if and only if $C(G)$ is contained in the centre of G.*

THEOREM 7. *$B_Y = G$ for every cyclic subgroup Y of G if and only if G is an L-group.*

PROOF. In order that $B_Y = G$ may hold for every cyclic sub-group Y of G, it is necessary and sufficient that $\big((a^m, b), a^n\big) = 1$ for every pair of elements a, b of G and all the exponents m, n. Putting $m = n = 1$, we obtain the necessary and sufficient condition for L-groups. On the other hand suppose G to be an L-group, then $(a^m, b) = (a, b^m)$ and this commutator must be commutative with a and therefore with a^n. Hence $\big((a^m, b), a^n\big) = 1$. Therefore $B_Y = G$ for every cyclic sub-group Y of G.

Suppose B_A is a group (e.g. if $A = G$); for every particular element b, the elements (b, a) form an Abelian group which is homomorphic to A. Simultaneously if a is kept constant and b runs over B_A, we obtain an Abelian group which is homomorphic to B_A. Groups with this kind of double homomorphism are necessarily Abelian. A thorough investigation will be given in a subsequent note.

ON $a^X - b^Y = b^y \pm a^x$

BY

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[Received 23 November 1943.]

1. In two previous papers*, I have proved that the equation $a^x - b^y = c$ has only a finite number of solutions for any c, and has at most one solution when c is large. These two results together mean that $a^X - b^Y = a^x - b^y$ has got only a finite number of solutions in x, y, X, Y.

The object of this paper is to prove that

$$a^X - b^Y = b^y \pm a^x$$

will have only a finite number of solutions in x, y, X, Y.

2. Notation. $n = b + M(a)$ means that $n - b$ is a multiple of a; $n = b + H(a^i)$ means that $n - b$ is divisible by a^i but not by a^{i+1}. $t = \pm 1$. If $b^y = za^x + t$ is possible, where $x \geqslant 2$ and $(z, a) = 1$, m, n stand for the smallest values of x, y resp. If $a^x = zb^y + t$ is possible, where $y \geqslant 2$ and $(z, b) = 1$, u, v stand for the smallest values of x, y resp. p always denotes a prime. $a = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $l = p_1^{\beta_1} \ldots p_r^{\beta_r}$; $T = p_1^{\alpha_1 i - \beta_1} \ldots p_r^{\alpha_r i - \beta_r}$.

3. Lemma 1. *If $d | a^i$, then $d | a^h$ when $h \geqslant d - 1$.*

If $p^\alpha | d$, then $p^\alpha | a^i$, and so $p | a$. Further $\alpha \leqslant d - 1$.

So $p^\alpha | p^{d-1}$ and $p^{d-1} | a^h$. Therefore $p^\alpha | a^h$. Hence the lemma.

Lemma 2. *If $kl = M(a^i)$ and $r \geqslant 2$, then*

$$\binom{k}{r} l^{r-1} a^{r-1} = M(a^i).$$

Let $kl = s . a^i$.

$$\binom{k}{r} = \binom{k-1}{r-1} \cdot \frac{k}{r} = \frac{kh}{r} = \frac{kh_1}{d_1}, \text{ where } d_1 = r/(h, r).$$

* *Jour. Ind. Math. Soc.* (2) 2 (1936-7), 119-22 and (1) 19 (1931-2), 1-11.

So $l\binom{k}{r} = \frac{lkh_1}{d_1} = \frac{sh_1a^i}{d_1} = \frac{ca^i}{d}$, where $d = d_1/(d_1, sh_1)$.

Since $l\binom{k}{r}$ is an integer, $d \mid a^i$. So, by Lemma 1 $d \mid a^h$, when $h \geqslant d-1$. But $d \mid r$, and so $d \leqslant r$.

Hence $d \mid a^{r-1}$. From this, the lemma follows.

Lemma 3. *When $s \geqslant 2$ and $h \geqslant 2$,*

$$\binom{T}{s} m_1^s l^s a^{sh} = M(a^{m+i+1}).$$

Now. $sh - h - (s-1) - 1 = (s-1)(h-1) - 1 \geqslant 0$.

So
$$a^{sh} = M(a^{h+1+s-1}).$$

By Lemma 2, $\binom{T}{s} l^s a^{s-1} = M(a^i)$.

The lemma now follows.

Lemma 4. *If $(a, b) = 1$, it is possible to find x, y such that $b^y = ka^x + 1$, where $x \geqslant 2$, and $(k, a) = 1$.*

Now the congruence $b^y \equiv 1 \pmod{a^2}$ is solvable. Let y_1 be the smallest value of y. Then $b^{y_1} = m_1 la^{x_1} + 1$, where $(m_1, a) = 1$, and $x_1 \geqslant 2$, and $\beta_s \leqslant \alpha_s - 1$ at least for one s.

Let $i = \text{Max} \left\{ [\beta_s/\alpha_s] + 1 \right\}$, $s = 1, 2, \ldots, r$.

Then $b^{Ty_1} = (1 + m_1 la^{x_1})^T = 1 + m_1 a^{x_1+i} + \sum_{s=2}^{T} \binom{T}{s} m_1^s l^s a^{sx_1}$

$= 1 + m_1 a^{x_1+i} + M(a^{x_1+i+1})$, by Lemma 3.

$= 1 + k.\, a^{x_1+i}$, where $(k, a) = 1$.

Lemma 5. *If a is odd and $b^y \equiv -1 \pmod{a}$ is solvable then we can find x_1 and y_1 such that $b^{y_1} = k\, a^{x_1} - 1$, where $(k, a) = 1$, and $x_1 \geqslant 2$.*

If $b^y = m_1 a - 1$, then $b^{ay} = -1 + M(a^2)$. Hence $b^y \equiv -1 \pmod{a^2}$ is solvable. The rest of the proof is similar to the previous one.

Lemma 6. *If λ is the smallest integer $\geqslant 1$ such that $b^\lambda \equiv t \pmod{a^\mu}$, where μ is given, and if $b^y \equiv t \pmod{a^\mu}$ then*

y is a multiple of λ when $t=1$, and an odd multiple of λ when $t=-1$.

Let $y=k\lambda+s$, where k is odd, and $0\leqslant s<2\lambda$. Now $b^{\lambda}\equiv t \pmod{a^{\mu}}$. So $b^{\lambda k}\equiv t \pmod{a^{\mu}}$. Further $b^{y}\equiv t \pmod{a^{\mu}}$. Hence $b^{y-\lambda k}\equiv 1 \pmod{a^{\mu}}$. i. e. $b^{s}\equiv 1 \pmod{a^{\mu}}$.

1. $t=1$. If $s\geqslant\lambda$, $b^{s-\lambda}\equiv 1 \pmod{a^{\mu}}$. Then $0\leqslant s-\lambda<\lambda$. So $s-\lambda=0$. If $s<\lambda$, then $s=0$.

2. $t=-1$. $b^{\lambda}\equiv -1 \pmod{a^{\mu}}$. But $b^{s}\equiv 1 \pmod{a^{\mu}}$. So $b^{|\lambda-s|}\equiv -1 \pmod{a^{\mu}}$. But, if $s\geqslant 1$, then $|\lambda-s|<\lambda$. This is against the definition of λ. So $s=0$. Hence y is an odd multiple of λ.

LEMMA 7. *Let a be odd when $t=-1$. If $b^{y}\equiv t \pmod{a^{x}}$, then $y=M(a^{x-m})$.*

By definition, $b^{n}=ka^{m}+t$. When $h=H(a^{i})$ and h is odd when $t=-1$, $b^{nh}=(t+ka^{m})^{h}=t+kha^{m}+M(a^{m+i+1})$, $=t+H(a^{m+i})$ by Lemma 3. So if $m+i=x$, $h=H(a^{x-m})$. Further from Lemma 6, $y=M(n)$. From these two, the lemma follows.

LEMMA 8. *When $a^{X}>b^{Y}$ and $(a,b)=1$, there is an X_0 such that for all $X\geqslant X_0$, $\frac{1}{2}(a^{X}-b^{Y})\geqslant a^{\frac{2}{3}X}>b^{\frac{2}{3}Y}$*

This follows from Theorem I in Reference (2).

THEOREM 1. *The equation* $a^{X}-b^{Y}=a^{x}+b^{y}$ (1) *has got only a finite number of solutions.*

Without loss of generality, we can assume that $X\geqslant x$, $Y\geqslant y$. If $(a,b)=d>1$, then Min $(X,Y)=$ Min (x,y); Min $(X,y)=$ Min (x,Y); Min $(X,x)=$ Min (Y,y). Hence three of the indices are equal, and consequently the fourth is also equal to the rest. So (1) has no solution. Hence we can assume that $(a,b)=1$. From (1) we get

$$a^{x}(a^{X-x}-1)=b^{y}(b^{Y-y}+1). \tag{2}$$

A. Let $X\geqslant X_0$.

(i) $a^{x}\geqslant b^{y}$. Then from Lemmas 8 and 1 we get

$$a^{x}\geqslant \tfrac{1}{2}(a^{X}-b^{Y})\geqslant b^{\frac{2}{3}Y}\geqslant 2^{\frac{2}{3}Y}>\tfrac{1}{10}Y^{2}. \tag{3}$$

If there is an r such that b^y+1 is not divisible by a^{r+1}, then there is no solution for (2) when $x \geqslant r+1$. In that case $x \leqslant r$ so that we can assume $b^y \geqslant a^x$ and so we need not consider it in (i). On the other hand, if there is no such r, by Lemma 5 the existence of m and n follows. Then from (2) and Lemma 7, $Y - y = M(a^{x-m})$ so that

$$Y \geqslant a^{x-m} \tag{4}$$

From (3) and (4),

$$a^x > \tfrac{1}{10} Y^2 > \tfrac{1}{10} a^{2(x-m)} \geqslant a^{2(x-m-2)}. \tag{5}$$

Hence $x > 2(x-m-2)$. But this is impossible when $x > 2(m+2)$. So (2) has no solution when

$$X \geqslant x_0,\ a^x \geqslant b^y \text{ and } x > 2(m+2). \tag{6}$$

(ii) $b^y > a^x$. Then from (1) and Lemma 8

$$b^y \geqslant \tfrac{1}{2}(a^X - b^Y) > a^{\frac{2}{3}X} \geqslant 2^{\frac{2}{3}X} > \tfrac{1}{10} X^2. \tag{7}$$

From Lemma 4, we can find x and y such that $a^x = kb^y + 1$, where $(k, b) = 1$ and $y \geqslant 2$, i.e. the existence of u and v follows. Then from (2) and Lemma 7, $X - x = M(b^{y-v})$ so that $X \geqslant b^{y-v}$. From (7) and (8), we have

$$b^y > \tfrac{1}{10} X^2 > \tfrac{1}{10} b^{2(y-v)} \geqslant b^{2(y-v-2)} \tag{8}$$

So $y > 2(y-v-2)$, which is impossible when $y > 2(v+2)$. Hence (2) has no solution when

$$y > 2(v+2),\ b^y > a^x \text{ and } X \geqslant X_0. \tag{9}$$

B. Let $X \leqslant X_0$. Then $a^X - b^Y \geqslant 0$ has only a finite number of solutions. Further, $a^x - b^y = c$ has only a finite number of solutions. Hence (1) has only a finite number of solutions with

$$X \leqslant X_0 \tag{10}$$

From (6), (9) and (10), the theorem follows.

THEOREM 2. *The equation* $a^X - b^Y = b^y - a^x$ *has only a finite number of solutions.*

The proof is similar to that of Theorem 1.

ON THE SMALLEST PRIMITIVE ROOT OF A PRIME.*

BY

S. S. PILLAI, (*Calcutta University*).

[Received 23 November 1943].

Let $g(p)$ be the smallest primitive root of a prime p. Vinogradow† has proved that

$$g(p) = O(p^{\frac{1}{2}+\varepsilon}) \text{ for every positive } \varepsilon.$$

The object of this note is to prove that

$$g(p) = \Omega\ (\log \log p). \qquad (1)$$

LEMMA 1. *If $p=4m+1$ be a prime, every divisor of m is a quadratic residue of p.*

Since 2 is a quadratic residue of every prime of the form $8n+1$, 2 is a quadratic residue of p, if m is even. (2)

Let d be an odd prime divisor of m. Then, by the law of quadratic reciprocity,

$$(d/p)(p/d) = (-1)^{(p-1)(d-1)/4} = 1,$$

where (p/q) denotes Legendre's symbol.

But $$(p/d) = \Big((rd+1)/d \Big) = (1/d) = 1.$$

So $$(d/p) = 1. \qquad (3)$$

Since the product of quadratic residues is again a quadratic residue, the lemma follows from (2) and (3).

LEMMA 2. *If k is any positive integer, there is at least one prime of the form $km+1$ not exceeding 2^k+1.*‡

Let $k = 2^r . p_1^{\alpha_1} \ldots p_l^{\alpha_l}$, where $r \geqslant 0$ and $\alpha \geqslant 1$.

(1) $x^{2^r}+1$ has got prime factors only of the form $2^{r+1}t+1$ besides 2.

* This was read at the conference of the Indian Mathematical Society held at Nagpur in 1929.

† Landau : *Vorlesungen uber Zahlentheorie* Kapital 14.

‡ More precise results are known.

(2) $(x^p+1)/(x+1)$ has got prime factors only of the form $mp+1$ besides p, where p is an odd prime.

(3) $(x^{p^a}+1)/(x^{p^{a-1}}+1)$ has got prime factors only of the form mp^a+1 besides p; and it is divisible by p only when $x \equiv -1 \pmod p$, and then not by p^2. (1) and (2) are well-known theorems of Euler, and (3) can be proved similarly. From these, it follows that

$2^k+1 = (2^{k/2^r})^{2^r}+1$ has got prime factors only of the form $m. 2^{r+1}+1$;

$(2^k+1)/(2^{k/p_1}+1)$has got prime factors only of the form $mp_1^{a_1}+1$, besides p_1;

$(2^k+1)/(2^{k/p_2}+1)$ has got prime factors only of the form $mp_2^{a_2}+1$, besides p_2; etc., and $(2^k+1)/(2^{k/p_l}+1)$ has got prime factors only of the form $mp_l^{a_l}+1$, besides p_l.

From the above we see that

$(2^k+1)/P$ has got prime factors only of the form $km+1$, besides $p_1,\ldots, p_l$, where P is the L. C. M. of $2^{k/p_1}+1, 2^{k/p_2}+1, \ldots\ldots, 2^{k/p_l}+1$. Therefore, if $Q=p_1^{a_1}\ldots p_l^{a_l}$, where a_s is 1 or zero according as $(2^k+1)/P$ is or is not divisible by p_s, it follows that $T=(2^k+1)/(PQ)$ has got prime factors only of the form $km+1$. Now

$$P = \Pi\,(x^t+1)\ \Pi\ 1/(x^{t'}+1),$$

where $x=2^{k/(p_1\cdots p_l)}$ and t runs through all the products of $p_1,\ldots, p_l$ taken $l-2s+1$ at a time $[1 \leqslant s \leqslant (l+1/2)]$, and t' runs through all the products of $p_1,\ldots, p_l$ taken $l-2s$ at a time $(1 \leqslant s \leqslant l/2)$. Now in the numerator of P, there are at most 2^{l-1} factors. Hence

$$\begin{aligned}\log T &= \log(2^k+1)-\log P-\log Q\\ &\geqslant k(1-1/p_1)\ldots(1-1/p_l)\log 2-2^{l-1}-\log Q\\ &\geqslant k(1-1/p_1)\ldots(1-1/p_l)\log 2-2^{l-1}-\log(p_1\ldots p_l).\end{aligned}$$

Now if $p_1,\ldots, p_l$ are in ascending order, then $p_1-1\geqslant 2$, $p_2-1\geqslant 4$, $p_3-1\geqslant 6$, etc. Therefore, when $l\geqslant 2$,

$$(p_1-1)\ldots(p_l-1)\geqslant 4.\ 2^{l-1}$$

Hence, when $l \geqslant 2$,

$$\log T \geqslant (p_1-1)\ldots(p_l-1)\ (\log 2-\tfrac{1}{4})-\log\ (p_1\ldots p_l)$$
$$\geqslant \tfrac{2}{5}\ (p_1-1)\ldots.(p_l-1)-\log\ (p_1\ldots p_l)$$
$$> 0.$$

So, when $l \geqslant 2$, $T > 1$. Hence, when $l \geqslant 2$, T has got at least one prime factor of the form $km+1$. (5)

When $l=1$, $(2^k+1)(2^{k/p_1}+1) > 2^{k(1-1/p_1)}$
$$\geqslant 1+k(1-1/p_1)+1 \geqslant p_1+1.$$

So when $l=1$, $(2^k+1)/\left\{ p_1(2^{k/p_1}+1) \right\} > 1$; further it has prime factors only of the form $km+1$. Hence, in this case also, the lemma is true. (6)

When $l=0$ it is obvious. (7)

From (5), (6), (7) the lemma follows.

Now we are in a position to prove our result.

THEOREM 1. *If q, the smallest prime of the form $2km+1$, satisfies the inequality $k > f(q)$, then,*

$$g(q) > (1-\delta)\log \left\{ f(q) \right\}.$$

Put $k=2.3\ldots p$, where $2, 3, \ldots, p$ are primes in order. Then, by hypothesis,

$$k > f(q), \tag{8}$$

since $\log k \sim p$. So

$p \sim \log k > \log \left\{ f(q) \right\}$. That means
$$p > (1-\delta)\log \left\{ f(q) \right\}, \text{ when } p \geqslant p_0. \tag{9}$$

Now for the prime q, there is no primitive root $\leqslant p$, for by Lemma 1, every prime $\leqslant p$ is a quadratic residue of q; hence every integer $\leqslant p$ is a quadratic residue of q. Therefore, from (9),

$$g(q) \geqslant p > (1-\delta)\log \left\{ f(q) \right\}.$$

THEOREM 2. *For an infinity of q's,*

$$g(q) > \frac{1-\delta}{\log 2}\log\log q.$$

Define q as in Theorem 1. Then from Lemma 2,

$$q \leqslant 2^k + 1. \tag{10}$$

That is, $k \geqslant \{\log (q-1)\}/\log 2.$

But, from Theorem 1,

$$\begin{aligned} g(q) &\geqslant (1-\delta) \log \{ f(q) \} \\ &\geqslant (1-\delta) \log \log q/\log 2. \end{aligned}$$

THEOREM 3.

$$g(q) \doteq \Omega (\log \log q).$$

This is a weaker form of Theorem 2.

THEOREM 4. *If the generalized Riemann hypothesis is true,*

$$g(p) \geqslant \{ (1-\delta) \log p \}/2$$

for infinitely many p.

ON WARING'S PROBLEM WITH POWERS OF PRIMES.

BY

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[Received 23 November 1943].

1. Let p^θ be the highest power of the prime p which divides the integer k, and let

$$\gamma = \begin{cases} \theta+2 \text{ when } p=2 \text{ and } 2|k \\ \theta+1 \text{ otherwise.} \end{cases}$$

$K = p_1^{\gamma_1} \ldots p_n^{\gamma_n}$, where $p_1, \ldots, p_n$ are primes such that $(p_r - 1)|k$, for $r = 1, 2, \ldots, n$. $d_r = p_r^{\gamma_r}$; $Q_r = d_1 d_2 \ldots d_r$; and $d_1 > d_2 > \ldots > d_n$. Hereafter p stands for a prime which divides K and P denotes a prime which does not divide K. $f(N, Q_r)$ is the least value of s such that $N \equiv x_1 + \ldots + x_s \pmod{Q_r}$. $F(Q_r) = \overline{\lim}\, f(N, Q_r)$ and $t(k) = F(Q_n) = F(K)$ where $x = 0$, p^k or P^k. $G_1(k)$ is the least value of s such that every *large* number is the sum of s or fewer k th powers of primes.

The object of this note is to prove

THEOREM I. $\qquad G_1(k) \geqslant t(k) + k + 1.$

THEOREM II. $\quad t(k) \leqslant d_1 + d_3 + d_4 + \ldots + d_n - n + 1.$

Theorems I and II are improvements on my previous results.

2. Let A be the least integer which requires $t(k)$ terms, i. e. $f(A, Q_n) = t(k)$. Further, let

$$N \equiv A + k + 1 \pmod{K}. \tag{1}$$

and s be the least s such that

$$N = x_1 + \ldots + x_s$$

where $x = 0, p^k, P^k$ for all large N satisfying (1). $\qquad$ (2)

If possible, let the number of x equal to P^k in (2) be less than or equal to k. Then if $N \leqslant X$, the number of N's

$$\leqslant (n+1)^{s-k} \left\{ \pi(X^{1/k}) \right\}^k = o(X),$$

where $\pi(X)$ denotes the number of primes not exceeding X.

Hence the number of x equal to P^k is $\geqslant k+1$.

But $P^k \equiv 1 \pmod{K}$. Therefore

$$\begin{aligned} x_1+\dots+x_{s-k-1}+k+1 &\equiv x_1+\dots+x_s \pmod{K} \\ &\equiv N \pmod{K} \\ &\equiv A+k+1 \pmod{K}, \end{aligned}$$

so that $A \equiv x_1+\dots+x_{s-k-1} \pmod{K}$. Since $f(A, K) = t(k)$, we get $s-k-1 \geqslant t(k)$. Since $G_1(k) \geqslant s$, we get Theorem I.

3. LEMMA 1. $p_r^k \equiv h_r K/d_r+1 \pmod{K}$, *where* $(h_r, d_r) = 1$.

LEMMA 2. $\sum\limits_{s=1}^{r} p_s^k \equiv r-1 \pmod{Q_r}$.

Lemma 1 is an easy consequence of Fermat's theorem. It is assumed that $k \geqslant 3$. If the modulus is Q_r, from Lemma 1 we have

$$0 \equiv \prod_1^r p_s^k \equiv \Pi\left\{h_s \cdot \frac{Q_r}{d_s}+1\right\} \equiv 1+\sum_1^r h_s \frac{Q_r}{d_s}$$

$$\equiv 1+\sum_1^r (p_s^k-1), \text{ which is Lemma 2.}$$

LEMMA 3. $F(Q_2) \leqslant d_1-1$.

Let $0 \leqslant a \leqslant d_1-1$ and A, B, C be the least positive residues of ap_1^k, $(d_2-a)\, p_2^k$ and $(d_1-d_2+a)\, p_1^k \pmod{Q_2}$. If the modulus is Q_2, since $\gamma_1 \leqslant k$,

$$A \equiv ap_1^k \equiv amp_1^{\gamma_1} = amd_1 \text{ and } a p_1^k \not\equiv 0,$$

so that

$$A \leqslant Q_2-d_1. \tag{3}$$

From Lemma 2, $p_1^k+p_2^k \equiv 1$.

$$B-A \equiv (d_2-a)\,(1-p_1^k)-ap_1^k \equiv -d_2 p_1^k+d_2-a \equiv d_2-a. \tag{4}$$

$$C-B=d_1p_1^k-d_2p_1^k+ap_1^k-d_2+a+d_2p_1^k-ap_1^k=d_1-d_2+a \tag{5}$$
$$C-A=C-B+B-A=d_1. \tag{6}$$
From (3)-(6), we get
$$0\leqslant A<B<C.$$

Let $A\leqslant N\leqslant B-1$. Then $N=ap_1^k+(N-A)P^k$. Let $B\leqslant N\leqslant C-1$. Then $N=(d_2-a)\,p_2^k+(N-B)\,P^k$. Now $a+N-A\leqslant a+B-1-A\leqslant d_2-1$ and $d_2-a+N-B\leqslant d_2-a+C-1-B\leqslant d_1-1$.

When a runs from 0 to d_2-1, the whole range will be covered. Hence the lemma.

LEMMA 4. *If $F(Q_r)\leqslant m$, then $F(Q_{r+1})\leqslant m+d_{r+1}-1$ where $r\leqslant n-1$.*

If N be any integer, by hypothesis,
$$N-d_{r+1}+1\equiv x_1+\ldots+x_m \pmod{Q_r}.$$
Hence $N-d_{r+1}+1\equiv lQ_r+x_1+\ldots+x_m \pmod{Q_{r+1}}$.

From Lemma 1, we can choose $a_{r+1}\leqslant d_{r+1}-1$ so that
$$a_{r+1}p_{r+1}^k\equiv a_{r+1}\,(h_{r+1}Q_r+1) \pmod{Q_{r+1}}$$
$$\equiv lQ_r+a_{r+1} \pmod{Q_{r+1}}.$$
Hence
$$N\equiv a_{r+1}p_{r+1}^k+x_1+\ldots+x_m+(d_{r+1}-1-a_{r+1})\,P^k \pmod{Q_{r+1}}.$$
Since N is arbitrary, we get the lemma.

Theorem II is an immediate consequence of Lemmas 3 and 4.

THEOREM III. $t(k)\geqslant d_1-1$.

If $(a,p_1)=1$, $a^k\equiv 1 \pmod{d_1}$ and $p_1^k\equiv 0 \pmod{d_1}$.

Hence if $N\equiv d_1-1 \pmod{d_1}$, then N requires at least d_1-1, kth powers of primes.

THEOREM IV. *If $K/d_1\geqslant d_1$, then $t(k)\geqslant d_1$.*

If $q_1,\ldots,q_l$ are prime to d_1 and $rd_1\equiv a_1q_1^k+\ldots+a_lq_l^k \pmod{d_1}$, then $a_1+\ldots+a_l\geqslant d_1$.

If $rd_1\equiv ap_1^k$ and $p_1^k=md_1$, then $am\equiv r \pmod{K/d_1}$

So for one $r\leqslant K/d_1-1$, $a=K/d_1-1$.

Hence $t(k)\geqslant \text{Min}\,\{d_1,\,K/d_1-1\}=d_1$.

INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS OF IMAGINARY ARGUMENT*

BY

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[Received 23 December, 1943].

The object of this paper is to evaluate the integral

$$I=\int_0^\infty x^{n-1}K_m(ax)\,{}_pF_q\left\{\alpha_1,\ldots,\alpha_p;\ \beta_1,\ \ldots,\ \beta_q;-b^2x^2\right\}dx. \qquad (1)$$

LEMMA 1. *When* $p-q\leqslant 1$.

$$I_1=\int_0^\infty x^{\delta-1}e^{-Ax^2}\,{}_pF_q(a_1,\ldots,\ a_p;\ b_1,\ldots,b_q;\ bx^2)dx$$

$$=\sum_0^\infty\frac{(a_1,\ r)\ldots(a_p,\ r)}{r\,!\,(b_1,\ r)\ldots(b_q,\ r)}\ b^r\int_0^\infty x^{\delta+2r-1}e^{-Ax^2}dx$$

$$=\frac{\Gamma(\delta/2)}{2A^{\delta/2}}\ {}_{p+1}F_q\left\{a_1,\ldots,\ a_p,\ \frac{\delta}{2};\ b_1,\ldots,\ b_q;\frac{b}{A}\right\}. \qquad (2)$$

LEMMA 2. *When* $R(s\pm m+\frac{1}{2})>0$,

$$I_2=\int_0^\infty u^{s-1}e^{-\frac{1}{2}u}W_{k,\,m}(u)\,{}_pF_q(a_1,\ldots,\ a_p;\ b_1,\ldots,\ b_q;\ -cu)du$$

$$=\sum_0^\infty\frac{(a_1,\ r)\ldots(a_p,\ r)\ (-c)^r}{r,\ (b_1,\ r)\ldots(b_q,\ r)}\int_0^\infty u^{s+r-1}e^{-\frac{1}{2}u}W_{k,\,m}(u)du$$

$$=\frac{\Gamma(s+m+\frac{1}{2})\ \Gamma(s-m+\frac{1}{2})}{\Gamma(s-k+1)}$$

$$\times\ {}_{p+2}F_{q+1}\left\{\begin{matrix}a_1,\ldots,\ a_p,\ s+m+\frac{1}{2},\ \ s-m+\frac{1}{2};\\ b_1,\ldots,\ b_q,\ s-k+1;\qquad\qquad -c\end{matrix}\right\} \qquad (3)$$

using a formula of Goldstein†.

* I wish to express my thanks to Dr. B. Mohan for his help in the preparation of this paper.

† *Proc. Lond. Math. Soc.* 34 (1932), 103—25.

Now, start with the integral

$$L=\int_0^\infty x^{2l+2k-2}K_{2m}(ax)\,{}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;-b^2x^2)\,dx$$

$$=\frac{2^{2k-2}}{a^{2k-1}}\int_0^\infty x^{2l+2k-2}\,{}_pF_q\left\{\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ -b^2x^2\right\}dx$$

$$\times\int_0^\infty \frac{e^{-(a^2x^2/4u)}\ e^{-\frac{1}{2}u}}{x^{2k-1}}u^{k-2}W_{k,m}(u)\,du,\ *$$

$$=\frac{2^{2k-2}}{a^{2k-1}}\int_0^\infty e^{-\frac{1}{2}u}u^{k-2}W_{k,m}(u)\,du\int_0^\infty x^{2l-1}e^{-\frac{a^2x^2}{4u}}$$

$$\times\,{}_pF_q\left\{\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ -b^2x^2\right\}dx.$$

The inversion of the order of integration is justifiable when the double integral

$$\int_0^\infty \left|e^{-\frac{1}{2}u}u^{k-2}W_{k,m}(u)\right|du\int_0^\infty \left|x^{2l-1}e^{-\frac{a^2x^2}{4u}}\right|\times$$

$$\left|{}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ -b^2x^2)\right|dx$$

is convergent. This is so if $R(n\pm m)>0$

(i) for all values of a and b when $p+2\leqslant q$;

and (ii) for $|b|<a/2$ when $p+1=q$.

Using Lemmas 1 and 2 and putting $l+k=\frac{1}{2}(n+1)$ we get, under the above conditions,

$$I=\frac{2^{n-2}}{a^n}\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right)$$

$$\times\,{}_{p+2}F_q\left\{\alpha_1,\ldots,\alpha_p,\ \frac{n+m}{2},\ \frac{n-m}{2};\ \beta_1,\ldots,\beta_q;\ -\frac{4b^2}{a^2}\right\}.\quad (4)$$

PARTICULAR CASES.

(i) When $b=0$, $R(n\pm m)>0$, we get Heaviside's result†

$$\int_0^\infty x^{n-1}K_m(ax)\,dx=\frac{2^{n-2}}{a^n}\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right).\qquad (5)$$

* Goldstein: *Ibid* § 3.4 (49).

† G. N. Watson: *Theory of Bessel Functions*, § 13.21 (8).

(ii) Putting $p=0$, $q=1$, $\beta_1=\nu+1$, $b^2=c^2/4$ and $n=l+\nu$, we get another result of Heaviside,*

$$\int_0^\infty x^{l-1}K_m(ax)J_\nu(cx)dx$$

$$=\frac{c^\nu 2^{l-2}\Gamma\left\{\tfrac{1}{2}(l+\nu+m)\right\}\Gamma\left\{\tfrac{1}{2}(l+\nu-m)\right\}}{a^{l+\nu}\Gamma(\nu+1)}$$

$$\times {}_2F_1\left\{\frac{l+\nu+m}{2},\frac{l+\nu-m}{2};\nu+1,-c^2/a^2\right\}, \qquad (6)$$

where $R(a)>R(b)$ and $R(l+\nu\pm m)>0$.

(iii) Put $p=1$, $q=2$, $b^2=c^2/4$, $\alpha_1=1$, $\beta_1=\tfrac{3}{2}$, $\beta_2=\nu+\tfrac{3}{2}$ and $n=l+\nu+1$. We get Mohan's formula†

$$\int_0^\infty x^{l-1}K_m(ax)H_\nu(cx)dx$$

$$=\frac{b^{\nu+1}2^{l-1}\Gamma\left(\frac{l+\nu+m+1}{2}\right)\Gamma\left(\frac{l+\nu-m+1}{2}\right)}{\sqrt{\pi}a^{l+\nu+1}\Gamma(\nu+\tfrac{3}{2})}$$

$$\times {}_3F_2\left\{1,\frac{l+\nu+m+1}{2},\frac{l+\nu-m+1}{2};\tfrac{3}{2},\nu+\tfrac{3}{2};-c^2/a^2\right\}, \qquad (7)$$

where $R(l+\nu\pm m+1)>0$ and $H_\nu(z)$ is Struve's function.

(iv) Put $p=0$, $q=3$, $b^2=c^2/6^4$, $\beta_1=p+\tfrac{1}{2}$, $\beta_2=p+1$, $\beta_3=2p+1$, and $n=l+2p$. We have

$$\int_0^\infty x^{l-1}K_m(ax)I_{2p}(\sqrt{cx})\,J_{2p}(\sqrt{cx})dx$$

$$=\frac{c^{2p}2^{l-2p-2}\Gamma\left(p+\frac{l+m}{2}\right)\Gamma\left(p+\frac{l-m}{2}\right)}{a^{l+2p}\left\{\Gamma(1-2p)\right\}^2}$$

$$\times {}_2F_3\left\{p+\frac{l+m}{2},p+\frac{l-m}{2};p+\tfrac{1}{2},p+1,2p+1;-\frac{c^2}{4\cdot 3^4a^2}\right\} \qquad (8)$$

where $R(2p+l\pm m)>0$ and $p>\tfrac{1}{2}$.

* *Ibid.* 13·45 (1). In the rest of the paper, details of the proofs are omitted, depending on the use of known formulae which wil be found in Watson, *loc. cit.*, or in Bailey, *Cambridge Tracts* No. 32.

† Mohan, *Quart. Jour. Math.* (Oxf.) 13 (1941), 40-47, § 2.

(v) (*a*) Put $\alpha_1 = \frac{1+\nu+\delta}{2}$, $\alpha_2 = \frac{2+\nu+\delta}{2}$, $\beta_1 = 1+\nu$, $\beta_2 = 1+\delta$, $\beta_3 = 1+\nu+\delta$, $p = 2$, $q = 3$, and $n = l+\nu+\delta$.

We get

$$\int_0^\infty x^{l-1}K_m(ax)J_\nu(bx)J_\delta(bx)dx$$

$$= \frac{2^{l-2}b^{\nu+\delta}\Gamma\left(\frac{l+\nu+\delta+m}{2}\right)\Gamma\left(\frac{l+\nu+\delta-m}{2}\right)}{a^{l+\nu+\delta}\Gamma(\nu+1)\Gamma(\delta+1)}$$

$$\times {}_4F_3\left\{\begin{matrix}\frac{1+\nu+\delta}{2}, \frac{2+\nu+\delta}{2}, \frac{l+\nu+\delta+m}{2}, \frac{l+\nu+\delta-m}{2}; \\ 1+\nu, 1+\delta, 1+\nu+\delta;\end{matrix} -\frac{4b^2}{a^2}\right\} \quad (9)$$

where $R(l+\nu+\delta\pm m) > 0$, $R(\delta) > -1$ and $R(\nu) > -1$.

With the same values as in (*a*), and with $-b^2$ for b^2, we get $\int_0^\infty x^{l-1}K_m(ax)I_\nu(bx)I_\delta(bx)dx$ = the expression on the right side of (1.9) after changing $-4b^2/a^2$ into $+4b^2/a^2$.

This is a particular case of a result given by Bailey* in terms of Appell's function.

(vi) If we put $b = a/2$, $\alpha_1 = \alpha$, $\alpha_2 = \alpha/2+1$, $\beta_1 = \alpha/2$, $\beta_2 = 1+\alpha-\frac{m+n}{2}$, $\beta_3 = 1+\alpha+\frac{m-n}{2}$, we get

$$\int_0^\infty x^{n-1}K_m(ax)$$

$$\times {}_2F_3\left(\alpha, \frac{\alpha}{2}+1; \frac{\alpha}{2}, 1+\alpha-\frac{m+n}{2}, 1+\alpha+\frac{m-n}{2}; -\frac{a^2x^2}{4}\right)dx$$

$$= \frac{2^{n-2}\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right)\Gamma\left(1+\alpha-\frac{m}{2}-\frac{n}{2}\right)}{a^n\Gamma(\alpha+1)}$$

$$\times \frac{\Gamma\left(1+\alpha+\frac{m}{2}-\frac{n}{2}\right)}{\Gamma(\alpha-n+1)}, \quad (10)$$

where $R(n\pm m) > 0$ and $\alpha+1 > n > 0$.

* *J. L. M. S.* 11, (1936), 16-20.

(vii) Let $p = 1$, $q = 2$, $\alpha_1 = \alpha$, $\beta_1 = 2\alpha$, $\beta_2 = \dfrac{n+1}{2}$, and $b^2 = -a^2/4$. We get

$$\int_0^\infty x^{n-1} K_m(ax)\, {}_1F_2\left(\alpha;\ 2\alpha, \frac{n+1}{2};\ \frac{a^2x^2}{4}\right)dx$$

$$= \frac{2^{2n-4}\Gamma(\alpha+\frac{1}{2})\,\Gamma\left(\alpha+\frac{1-n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+m}{4}\right)\Gamma\left(\frac{n-m}{4}\right)}{\sqrt{\pi}\,a^n\,\Gamma\left(\alpha+\frac{1}{2}-\frac{n+m}{2}\right)\Gamma\left(\alpha+\frac{1}{2}+\frac{m-n}{2}\right)} \tag{11}$$

where $R(n \pm m) > 0$, $\alpha > -\frac{1}{2}$ and $n > -\frac{1}{2}$.

(viii) Put $p = 2$, $q = 3$, $\alpha_1 = \alpha$, $\alpha_2 = \beta - \alpha$, $\beta_1 = \beta$, $\beta_2 = \dfrac{\beta}{2}$, $\beta_3 = \dfrac{\beta+1}{2}$ and $-b^2$ for b^2. We have

$$\int_0^\infty x^{n-1} e^{2bx} K_m(ax)\, {}_1F_1(\alpha;\ \beta;\ 2bx)\, {}_1F_1(\beta-\alpha;\ \beta;\ 2bx)\,dx$$

$$= \frac{2^{n-2}}{a^n}\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right)$$

$$\times\, {}_4F_3\left\{\begin{matrix}\alpha, \beta-\alpha, \dfrac{m+n}{2}, \dfrac{n-m}{2}; \\ \beta, \dfrac{\beta}{2}, \dfrac{\beta+1}{2}; \end{matrix}\ \frac{4b^2}{a^2}\right\}, \tag{12}$$

where $R(n \pm m) > 0$. In particular, if $\beta = 2\alpha = 2p+1$, and $n = l + 2p$, we get

(*a*)
$$\int_0^\infty x^{l-1} e^{4bx} K_m(ax) \left\{ I_p(bx) \right\}^2 dx$$

$$= \frac{2^{l-2} b^{2p}\, \Gamma\left(p+\frac{l+m}{2}\right)\Gamma\left(p+\frac{l-m}{2}\right)}{a^{l+2p}\ \left\{\Gamma(p+1)\right\}^2}$$

$$\times\, {}_3F_2\left\{p+\tfrac{1}{2},\ p+\frac{l+m}{2},\ p+\frac{l-m}{2};\ 2p+1,\ p+1;\ \frac{4b^2}{a^2}\right\}, \tag{13}$$

where $R\left(p+\dfrac{l \pm m}{2}\right) > 0$ and $p + \frac{1}{2} > 0$.

In particular, when $l=1$, $a=2b$, we get

$$\int_0^\infty e^{4bx}K_m(2bx)\left\{I_p(bx)\right\}^2 dx$$

$$=\frac{\Gamma\left(\frac{p}{2}+\frac{m}{4}+\frac{1}{4}\right)\Gamma\left(\frac{p}{2}-\frac{m}{4}+\frac{1}{4}\right)}{2^3 b\Gamma\left(\frac{p}{2}+\frac{m}{4}+1\right)\Gamma\left(\frac{p}{2}-\frac{m}{4}+1\right)}, \tag{14}$$

where $R(2p+1\pm m)>0$.

Again, putting $\alpha=l-k+\frac{1}{2}$, $\beta=2l+1$, $n=p+2l+1$ and $-b^2/4$ for b^2, we have

(*b*)
$$\int_0^\infty x^{p-1}e^{3bx}K_m(ax)M'_{k,l}(bx)M_{-k,l}(bx)dx$$

$$=\frac{2^{p+2l-1}b^{2l+1}}{a^{p+2l+1}}\Gamma\left(\frac{1+p+m}{2}+l\right)\Gamma\left(\frac{1+p-m}{2}+l\right)$$

$$\times{}_4F_3\left\{l-k+\tfrac{1}{2},\ l+k+\tfrac{1}{2},\ \frac{1+p+m}{2}+l,\ \frac{1+p-m}{2}+l;\ \frac{b^2}{a^2}\right\}, \tag{15}$$

where $R(p+2l+1\pm m)>0$.

In the same way we can derive other interesting integral formulae involving Sonine's polynomials, Bateman's functions, Weber's parabolic cylinder functions and Hermite polynomials in different combinations.

THE DIFFERENTIAL EQUATIONS SATISFIED BY CERTAIN FUNCTIONS.

BY

A. M. MIAN AND S. CHOWLA.

[Received 26 February, 1944]

Let $p(n)$ denote the number of partitions of n. We prove

THEOREM. *The function* $y = \sum_{1}^{\infty} p(n) x^n, \quad (-1 < x < 1)$ *satisfies an algebraic differential equation.*

There are similar results when the coefficient $p(n)$ is replaced by $\sigma(n)$, the sum of the divisors of n, or by $t(n)$, Ramanujan's function, defined as the coefficient of x^n in the expansion of

$$x \left\{ (1-x)(1-x^2)(1-x^3)\ldots \right\}^{24}$$

PROOF. We write,

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}, \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 \sin^2 \phi}\, d\phi,$$
$$k^2 + k'^2 = 1,$$

while K′, E' are the same functions of k' that K, E are of k. Further,

$$q = \exp(-\pi K'/K).$$

It is known that

$$\frac{dE}{dk} = \frac{E-K}{k} \tag{1}$$

$$\frac{dK}{dk} = \frac{E-k'^2 K}{kk'^2} \tag{2}$$

$$\frac{dq}{dk} = \frac{\pi^2 q}{2kk'^2 K^2} * \tag{3}$$

* *See* p. 32 of Ramanujan's *Collected Papers*. Whittaker and Watson: *Modern Analysis*, 1927, Example 10, p. 488.

$$\prod_{n=1}^{\infty}(1-q^n)=2^{\frac{1}{3}}\left(\frac{K}{\pi}\right)^{\frac{1}{2}}q^{-\frac{1}{24}}k^{\frac{1}{12}}k'^{\frac{1}{3}} \tag{4}$$

$$Q=\sum_{0}^{\infty}p(n)q^n=\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right) \tag{5}$$

From (1) and (5) it follows that

$$Q,\ \frac{dQ}{dq},\ \frac{d^2Q}{dq^2},\ \frac{d^3Q}{dq^3} \tag{6}$$

are explicit algebraic functions of

$$q,\ k,\ K,\ E \tag{7}$$

Eliminating k, K, E from these four relations, we get

$$F(q, Q, Q_1, Q_2, Q_3)=0,$$

where $F(a, b, c, d, e)$ denotes a polynomial in a, b, c, d, e, while Q_m denotes the mth differential coefficient of Q with respect to q.

ON A THEOREM OF ALGEBRA AND ITS GEOMETRICAL APPLICATION

BY

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[Received 28 April, 1944].

The following theorem of algebra seems to be of interest:

THEOREM. *Let*

$$\Phi_\alpha = \sum_{i,j=1}^{n} a_{\alpha ij} x_i x_j, \quad a_{\alpha ij} = a_{\alpha ji}, \quad \alpha = 1, \ldots, r, \tag{1}$$

be r linearly independent quadratic forms in the n independent variables x_i[1]. *We set*

$$\frac{1}{2}\frac{\partial \Phi_\alpha}{\partial x_i} = \sum_{j=1}^{n} a_{\alpha ij} x_j = y_{i\alpha} \tag{2}$$

and consider the exterior quadratic forms[2]

$$\pi_{ij} = \sum_{\alpha=1}^{r} [y_{i\alpha} y_{j\alpha}]. \tag{3}$$

If among the nr linear forms $y_{i\alpha}$ *there are 3r independent ones, say* $y_{1\alpha}, y_{2\alpha}, y_{3\alpha}$, *then the forms* π_{ij} *determine* Φ_α *up to an orthogonal transformation.*

It is to be remarked that π_{ij} remain unaltered when Φ_α undergo an orthogonal transformation. Hence π_{ij} can determine Φ_α only up to an orthogonal transformation.

To prove the theorem we take another set of r quadratic forms

$$\bar{\Phi}_\alpha = \sum_{i,j=1}^{n} \bar{a}_{\alpha ij} x_i x_j, \quad \bar{a}_{\alpha ij} = \bar{a}_{\alpha ji}, \tag{1'}$$

1. We assume, unless otherwise stated, the following ranges of the indices employed: $i, j, k = 1, \ldots, n$: $\alpha, \beta, \gamma = 1, \ldots, r$.

2. For definitions and notations see E. Cartan, *Lecons sur les invariants integraux*, Paris 1922.

concerning which we make no assumption about their linear independence. We put

$$\frac{1}{2}\frac{\partial \bar{\Phi}_\alpha}{\partial x_i} = \sum_j \bar{a}_{\alpha ij} x_j = \bar{y}_{i\alpha} \tag{2'}$$

and suppose that

$$\sum_\alpha [y_{i\alpha} y_{j\alpha}] = \sum_\alpha [\bar{y}_{i\alpha} \bar{y}_{j\alpha}]. \tag{4}$$

The theorem is equivalent to saying that the condition (4) implies that Φ_α and $\bar{\Phi}_\alpha$ are related by an orthogonal transformation. The special case $r=1$ of the theorem is equivalent to the following statement, which is well known:

Let $$A = (a_{ij}), \quad \bar{A} = (\bar{a}_{ij})$$

be two symmetric matrices whose corresponding two-rowed determinants are equal. If one of the matrices is of rank $\geqslant 3$, then $\bar{A} = \pm A$.

LEMMA. *Let y_α, z_α; $\bar{y}_\alpha$, $\bar{z}_\alpha$ be two sets of 2r linear forms each, of which y_α, z_α are linearly independent, such that the equation*

$$\sum_\alpha [y_\alpha z_\alpha] = \sum_\alpha [\bar{y}_\alpha \bar{z}_\alpha] \tag{5}$$

holds. Then $\bar{y}_\alpha$, $\bar{z}_\alpha$ are linear combinations of y_β, z_β and are linearly independent.

That $\bar{y}_\alpha$, $\bar{z}_\alpha$ are linearly independent follows from the fact that $2r$ is the minimum number of linear forms in terms of which the exterior quadratic form in the left side of (5) can be expressed and that this number remains unaltered under linear transformations.

We now proceed to prove the lemma indirectly. Suppose that y_α, z_α, $\bar{y}_1$ are linearly independent. Among the $4r$ forms y_α, z_α, $\bar{y}_\alpha$, $\bar{z}_\alpha$ we take a base

$$y_\alpha, \ z_\alpha, \ \bar{y}_1, \ \bar{y}_{\alpha_1}, \ldots, \ \bar{y}_{\alpha_p}, \ \bar{z}_{\beta_1}, \ldots, \ \bar{z}_{\beta_q},$$

which are linearly independent and are such that every other form is a linear combination of them. We choose the independent variables such that

$$\bar{y}_{\alpha_1} = \ldots = \bar{y}_{\alpha_p} = \bar{z}_{\beta_1} = \ldots = \bar{z}_{\beta_q} = 0. \text{ ; if } \beta_l \neq 1,$$
$$y_\alpha \neq 0,\ z_\alpha \neq 0,\ \bar{y}_1 \neq 0.$$

By a renumbering of $\bar{y}_\alpha$, $\bar{z}_\alpha$ we can write (5) in the form

$$[y_1 z_1] + \ldots + [y_r z_r] = [\bar{y}_1 \bar{z}_1] + \ldots + [\bar{y}_s \bar{z}_s].$$

According as $\bar{z}_1$ occurs in the base or not, we shall divide the discussion into two cases:

Case 1. $\bar{z}_1$ does not occur in the base. Then we have

$$\bar{y}_\rho = \sum_\alpha (A_{\rho\alpha} y_\alpha + B_{\rho\alpha} z_\alpha) + C_\rho \bar{y}_1,$$
$$\bar{z}_\rho = \sum_\alpha (A'_{\rho\alpha} y_\alpha + B'_{\rho\alpha} z_\alpha) + C'_\rho \bar{y}_1,$$
$$\bar{z}_1 = \sum_\alpha (A_\alpha y_\alpha + B_\alpha z_\alpha) + C \bar{y}_1,\ \rho = 2, \ldots, s.$$

Substituting these expressions into the last equation and equating to zero the coefficients of the terms in $\bar{y}_1$, we get

$$A_\alpha - \sum_\rho C'_\rho A_{\rho\alpha} + \sum_\rho C_\rho A'_{\rho\alpha} = 0,\ B_\alpha - \sum_\rho C'_\rho B_{\rho\alpha} + \sum_\rho C_\rho B'_{\rho\alpha} = 0.$$

Using these relations, we find

$$\bar{z}_1 - \sum_\rho C'_\rho \bar{y}_\rho + \sum_\rho C_\rho \bar{z}_\rho = C \bar{y}_1,$$

which holds under the assumption that some of the $\bar{y}$'s and $\bar{z}$'s are zero. But this means that $\bar{y}_\alpha, \bar{z}_\alpha$ are linearly dependent and is therefore a contradiction.

Case 2. $\bar{z}_1$ belongs to the base. Then we have

$$\bar{y}_\rho = \sum_\alpha (A_{\rho\alpha} y_\alpha + B_{\rho\alpha} z_\alpha) + C_\rho \bar{y}_1 + D_\rho \bar{z}_1,$$
$$\bar{z}_\rho = \sum_\alpha (A'_{\rho\alpha} y_\alpha + B'_{\rho\alpha} z_\alpha) + C'_\rho \bar{y}_1 + D'_\rho \bar{z}_1,\ \rho = 2, \ldots, s.$$

Substituting into the equation, we get

$$\sum_\rho C'_\rho A_{\rho\alpha} - \sum_\rho C_\rho A'_{\rho\alpha} = 0,\ \sum_\rho C'_\rho B_{\rho\alpha} - \sum_\rho C_\rho B'_{\rho\alpha} = 0,$$
$$\sum_\rho (C_\rho D'_\rho - C'_\rho D_\rho) = -1,$$

and therefore

$$\sum_{\rho} (C'_{\rho} \bar{y}_{\rho} - C_{\rho} \bar{z}_{\rho}) = \sum_{\rho} (C'_{\rho} D_{\rho} - C_{\rho} D'_{\rho}) \bar{z}_1 = \bar{z}_1.$$

For the same reason as in Case 1 this leads to a contradiction. Our lemma is therefore proved.

The theorem now follows easily from the lemma. In fact, from (4) we have

$$\sum_{\alpha} [y_{r\alpha} y_{s\alpha}] = \sum_{\alpha} [\bar{y}_{r\alpha} \bar{y}_{s\alpha}], \quad r, s = 1, 2, 3, \; r \neq s \tag{6}$$

where $y_{1\alpha}, y_{2\alpha}, y_{3\alpha}$ are by assumption linearly independent. Applying the lemma successively to each of these equations, we see that $\bar{y}_{\lambda\alpha}$, for a fixed $\lambda = 1, 2, 3$, are linear combinations of $y_{\lambda\beta}$. We can therefore write

$$\bar{y}_{1\alpha} = \sum_{\beta} \lambda_{\alpha\beta} y_{1\beta}, \; \bar{y}_{2\alpha} = \sum_{\beta} \mu_{\alpha\beta} y_{2\beta}, \quad \bar{y}_{3\alpha} = \sum_{\beta} \nu_{\alpha\beta} y_{3\beta}, \tag{7}$$

where the matrices $(\lambda_{\alpha\beta})$, $(\mu_{\alpha\beta})$, $(\nu_{\alpha\beta})$ are to be denoted by L, M, N respectively. From (6) and (7) we easily get

$$L = M'^{-1}, \; M = N'^{-1}, \; N = L'^{-1}.$$

It follows that

$$L = M = N, \quad L' = L^{-1}.$$

The last equation shows that L is an orthogonal matrix.

The theorem is proved if $n = 3$. If $n \geqslant 4$, we suppose this orthogonal transformation to be performed, so that we have $\bar{y}_{1\alpha} = y_{1\alpha}$, $\bar{y}_{2\alpha} = y_{2\alpha}$, $\bar{y}_{3\alpha} = y_{3\alpha}$.

Equations (4) then give

$$\sum_{\alpha} [y_{1\alpha} (\bar{y}_{p\alpha} - y_{p\alpha})] = 0, \; \sum_{\alpha} [y_{2\alpha} (\bar{y}_{p\alpha} - y_{p\alpha})] = 0, \; p = 4, \ldots, n,$$

which shows that $\bar{y}_{p\alpha} - y_{p\alpha}$ are linear combinations both of $y_{1\beta}$ and of $y_{2\beta}$. Since $y_{1\beta}, y_{2\beta}$ are linearly independent, this is possible only when $\bar{y}_{p\alpha} - y_{p\alpha} = 0$.

Thus our theorem is proved.

The theorem in question has an application to the so-called rigidity problem of varieties in Euclidean space of n dimensions. Let V^m be a variety of m dimensions in a Euclidean space E^n. The metric in E^n induces on V^m a Riemannian metric. The problem is to see how

far V^m is determined by this Riemannian metric. We shall see that under certain supplementary conditions the variety is determined up to a motion or reflection.

To explain the result precisely, we give at first a few notions of differential geometry from the point of view of the method of moving frames.[3] In E^n we take as frame the figure $M\mathbf{n}_1\ldots\mathbf{n}_n$, where M is a point and $\mathbf{n}_1,\ldots,\mathbf{n}_n$ are n mutually perpendicular unit vectors through M forming an n-leg of a definite orientation. When a family of frames is given, we have a system of equations of the form

$$dM = \sum_i \omega_i \mathbf{n}_i, \quad d\mathbf{n}_i = \sum_i \omega_{ij}\mathbf{n}_j, \tag{8}$$

where $\omega_i = dM.\ \mathbf{n}_i,\ \omega_{ij} = -\omega_{ji} = d\mathbf{n}_i,\ \mathbf{n}_j$

are Pfaffian forms in the parameters of the family and are called the relative components. Two families of frames, depending on the same number of parameters, differ from each other only by a motion, when and only when a correspondence can be set between them such that the relative components are respectively equal. The relative components cannot be arbitrarily assigned and are bound by the following "equations of structure" of the space:

$$\omega_i' = \sum_j [\omega_j \omega_{ji}], \quad \omega'_{ij} = -\sum_k [\omega_{ik}\omega_{jk}]. \tag{10}$$

To apply the method of moving frames to the differential geometry of $V^m \subset E^n$ we attach to each point $M \subset V^m$ a frame $M\mathbf{n}_1,\ldots\mathbf{n}_n$ such that $M\mathbf{n}_1\ldots\mathbf{n}_m$ determine the tangent m-plane at M. Then we have

$$\omega_\lambda = 0, \quad \omega_{p\lambda} = \sum_q A_{q\lambda p}\omega_q, \quad A_{q\lambda p} = A_{p\lambda q}. \tag{11}$$

where, and also in later discussions, the ranges of the indices are as follows:

$$\lambda, \mu, \nu = m+1, \ldots, n, \quad p, q, r = 1, \ldots, m. \tag{12}$$

The $n-m$ quadratic differential forms

$$\Phi_\lambda = \sum_{p,q} A_{p\lambda q}\omega_p\omega_q \tag{13}$$

3. A good account of the geometry of subspaces in an Euclidean space of n dimensions is to be found in E. Cartan, *Bull. Soc. Math. France*, 47 (1919), 125-60; 48 (1920), 132—208.

can be called the second fundamental form of V^m. Among them let there be s linearly independent ones. Then we can choose the frame $M\mathbf{n}_1\ldots\mathbf{n}_n$ such that $\Phi_{m+1},\ldots,\Phi_{m+s}$ are linearly independent and $\Phi_{m+s+1}=\ldots=\Phi_n=0$. Under this condition the space $M\mathbf{n}_{m+1}\ldots\mathbf{n}_{m+s}$ is called the first normal space of V^m. The dimension of the first normal space is an arithmetic invariant of V^m. With this choice of the frame we have evidently

$$\omega_\lambda=0,\ \omega_{p,\ m+s+1}=\cdots=\omega_{pn}=0. \tag{14}$$

After this normalization of the second fundamental forms let us consider the linear forms

$$\tfrac{1}{2}\partial\Phi_t/\partial\omega_p=\omega_{pt},\quad t=m+1,\ .\ ,m+s.$$

If, in the table

$$\begin{array}{c}\omega_{1\,m+1},\ldots,\omega_{1\,m+s},\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \omega_{m\,m+1},\ldots,\omega_{m\,m+s}\end{array}$$

there are τ rows of ω's which are linearly independent while the ω's of any $\tau+1$ rows are linearly dependent, we say that V^m is of type τ at M. It is evident that $\tau\leqslant[m/s]$.

With these preparations we are going to give a proof of the following theorem, which was first established by Allendoerfer:[4]

THEOREM. *Let V^m and $\bar{V}^m$ be two varieties of m dimensions in E^n, whose induced Riemannian metrics are the same and whose first normal spaces have the same dimension. If one of them is of type $\geqslant 3$, the two varieties differ from each other only by a motion or a reflection.*

We suppose a family of frames $\bar{M}\bar{\mathbf{n}}_1\ldots\bar{\mathbf{n}}_n$ as described above is attached to the points of $\bar{V}^m$ and we shall denote without further explanation the corresponding quantities thus arisen by dashes. Then we have

$$\omega_\lambda=\bar{\omega}_\lambda=0,\quad \omega_{p\sigma}=\bar{\omega}_{p\sigma}=0, \tag{15}$$

in which, and also in what follows, we take

$$t,\ u=m+1,\ldots,m+s,\quad \rho,\ \sigma=m+s+1,\ldots,n. \tag{16}$$

4. C. B. Allendoerfer: Rigidity for spaces of class greater than one, *Amer. J. Math.* 61 (1939), 633-44.

The theorem will be proved, if we show that the frames $\bar{M}\bar{\mathbf{n}}_1\ldots\bar{\mathbf{n}}_n$ can be so chosen that the corresponding relative components are equal.

At first it is evident that, without destroying the existing conditions, we can rotate the vectors $\bar{\mathbf{n}}_1,\ldots,\bar{\mathbf{n}}_m$, such that we shall have

$$\bar{\omega}_p = \omega_p. \tag{17}$$

Taking the exterior derivative of (17) and making use of the equations of structure (10), we get

$$\sum_q[\omega_q(\bar{\omega}_{qp}-\omega_{qp})] = 0,$$

which gives $\bar{\omega}_{qp}-\omega_{qp} = P_{pqr}\omega_r$, where $P_{pqr} = P_{prq}$. But P_{pqr} are skew-symmetric in the indices q, p, since both ω_{qp} and $\bar{\omega}_{qp}$ are. It follows that $P_{pqr}=0$, or

$$\bar{\omega}_{qp} = \omega_{qp}. \tag{18}$$

The essential point of our proof is to establish

$$\bar{\omega}_{pt} = \omega_{pt}. \tag{19}$$

For this, we take the exterior derivative of (18), getting

$$\sum_t[\bar{\omega}_{pt}\bar{\omega}_{qt}] = \sum_t[\omega_{pt}\omega_{qt}]. \tag{20}$$

Our theorem on algebra is here applicable, and we conclude that Φ_t and $\bar{\Phi}_t$, or that $\bar{\omega}_{pt}$ and ω_{pt}, for every fixed p, differ from each other by an orthogonal transformation. Noticing that

$$\omega_{pt} = d\mathbf{n}_p.\mathbf{n}_t,\ \bar{\omega}_{pt} = d\bar{\mathbf{n}}_p.\bar{\mathbf{n}}_t,$$

we see that a suitable change of the vectors $\bar{\mathbf{n}}_t$, which leaves the conditions on our frames unaltered, will lead to the conditions (19). Now, exterior differentiations of the second set of equations (15) and of (19) give respectively

$$\sum_t[\omega_{pt}\omega_{\sigma t}] = \sum_t[\bar{\omega}_{pt}\bar{\omega}_{\sigma t}] = 0,\quad \sum_u[\bar{\omega}_{pu}\bar{\omega}_{tu}] = \sum_u[\omega_{pu}\omega_{tu}].$$

Since there are three sets of ω_{pt}, say ω_{1t}, ω_{2t}, ω_{3t}, which are linearly independent, we conclude easily that

$$\omega_{\sigma t} = \bar{\omega}_{\sigma t} = 0,\quad \bar{\omega}_{tu} = \omega_{tu}. \tag{21}$$

The conditions (15), (17), (18), (19), (21) express the fact that the relative components of the two families of frames are respectively equal except possibly $\omega_{\rho\sigma}$ and $\bar{\omega}_{\rho\sigma}$. But this can easily be achieved by an orthogonal transformation of the vectors $\bar{n}_\rho$. Hence we arrive at the conclusion that we can attach to each of the two varieties V^m and $\bar{V}^m$ a family of frames, one at each point, such that the relative components are respectively equal. The motion which carries one family of frames into another carries V^m into $\bar{V}^m$. That V^m and $\bar{V}^m$ possibly differ from each other by a reflection is due to the fact that we have applied orthogonal transformations on the vectors which may be improper. Therefore our theorem is proved.

The advantage of the present proof lies probably in the fact that it is simpler and that it brings out more clearly the algebraic nature of the problem.

ON 2-WEBS OF CURVES IN R_n.

BY

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[Received 19 February, 1944].

It has been shown* that every 2-web of curves in the affine space R_3 can be associated by means of projection to a differential equation of the second order $y'' = f(x, y, y')$. The object of this note is to show that certain 2-webs of curves in the affine space R_n can also be associated by means of projection to a differential equation of the $(n-1)$th order $y^{(n-1)} = f(x, y, y', \ldots, y^{(n-2)})$.

Suppose that two pencils of curves are given as the path curves of the two linear differentiators

$$\Delta_1 = \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \ldots + u_n \frac{\partial}{\partial u_{n-1}} + a_{n+1} \frac{\partial}{\partial u_n}, \text{ and } \Delta_2 = \frac{\partial}{\partial u_n}$$

* W. Blaschke: *Jber. dtsch. Math. Ver.* 38 (1929) 193-206.

Then we take $n-2$ other differentiators given by the following scheme:

$$\Delta_{i+1} = [\Delta_i, \ \Delta_1] = \Delta_i\Delta_1 - \Delta_1\Delta_i.$$

$$\Delta_3 = [\Delta_2, \Delta_1] = \frac{\partial}{\partial u_{n-1}} + a_{n+1,\,2}\frac{\partial}{\partial u_n}$$

$$\Delta_4 = \frac{\partial}{\partial u_{n-2}} + a_{n+1,\,2}\frac{\partial}{\partial u_{n-1}} + (a_{n+1,\,3} - a_{n+1,\,21})\frac{\partial}{\partial u_n}$$

$$\Delta_5 = \frac{\partial}{\partial u_{n-3}} + a_{n+1,\,2}\frac{\partial}{\partial u_{n-2}} + (a_{n+1,\,3} - 2a_{n+1,\,21})\frac{\partial}{\partial u_{n-1}}$$

$$+ (a_{n+1,\,4} - a_{n+1,\,31} + a_{n+1,\,213})\frac{\partial}{\partial u_n}$$

$$\Delta_n = \frac{\partial}{\partial u_2} + a_{n+1,\,2}\frac{\partial}{\partial u_3} + \ldots,$$

where $\Delta_i\Delta_j\Delta_k a_{n+1}$ has been denoted as $a_{n+1,\,kji}$. The determinant of these n differentiators is

$$\begin{vmatrix} 1 & u_3 & u_4 & u_5, \ldots & u_{n-1} & u_n & a_{n+1} \\ 0 & 0 & 0 & 0 \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \ldots & 0 & 1 & a_{n+1,\,2} \\ 0 & 0 & 0 & 0 \ldots & 1 & a_{n+1,\,2} & * \\ \cdot & \cdot & \cdot & \cdot \ldots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \ldots & \cdot & \cdot & \cdot \\ 0 & 1 & a_{n+1,\,2} & * \ldots & * & * & * \end{vmatrix} \neq 0.$$

Hence these n differentiators are linearly independent.

Also we have

$$\frac{\partial}{\partial u_n} = \Delta_2, \ \frac{\partial}{\partial u_{n-1}} = \Delta_3 - a_{n+1,\,2}\,\Delta_2,$$

$$\frac{\partial}{\partial u_{n-2}} = \Delta_4 - a_{n+1,2}\Delta_3 + (a^2_{n+1,2} + a_{n+1.3} - a_{n+1,21})\Delta_2, \text{ etc.},$$

from which it easily follows that $[\Delta_{i+1}, \Delta_i]$ is dependent linearly on $\Delta_2, \Delta_3, \ldots, \Delta_i$.

2. Conversely, suppose that two pencils of curves are given as the path curves of the two differentiators

$$\Delta_1 = \frac{\partial}{\partial u_1} + a_3\frac{\partial}{\partial u_2} + \ldots a_{n+1}\frac{\partial}{\partial u_n}, \ \Delta_2 = \frac{\partial}{\partial u_1} + b_3\frac{\partial}{\partial u_2} + \ldots + b_{n+1}\frac{\partial}{\partial u_n}.$$

These forms are permissible as the two differentiators —which are the same except for a common factor—define the same pencil as their path curves.

Now by a topological transformation

$$v_1 = f_1(u_1, \ldots, u_n), \ \ldots, v_n = f_n(u_1, \ldots, u_n),$$

a linear differentiator $\triangle$ is changed to a new differentiator $\triangle' = \triangle v_1 \,\partial/\partial v_1 + \cdots + \triangle v_n \,\partial/\partial v_n$. We shall say that $\triangle \equiv \triangle'$. Choose $v_1, \ldots, v_{n-1}$ as the solution of $\triangle f = 0$. Then $\triangle' = \triangle v_n . \,\partial/\partial v_n$. As $\triangle'$ and $\partial/\partial v_n$ differ by a factor $\triangle v_n$, they have the same path curves and therefore $\triangle \equiv \partial/\partial v_n$.

The path curves of these differentiators are given by the intersection of $n-1$ cylindrical surfaces $f_1(v_1, \ldots, v_{n-1}) =$ const. , $f_2(v_1, \ldots, v_{n-1}) =$ const, ..., $f_{n-1}(v_1, \ldots, v_{n-1}) =$ const. In particular they may also be given by the intersection of the surfaces $v_1 =$ const, ..., $v_{n-1} =$ const.

Therefore $\triangle_1$ and $\triangle_2$ can be taken by a suitable topological transformation to the form

$$\triangle_1 \equiv \frac{\partial}{\partial u_1} + a_3 \frac{\partial}{\partial u_2} + \cdots + a_{n+1} \frac{\partial}{\partial u_n}, \ \triangle_2 \equiv \frac{\partial}{\partial u_n}.$$

Thus

$$\triangle_3 = [\triangle_2 \triangle_1] = a_{3,2} \frac{\partial}{\partial u_2} + a_{4,2} \frac{\partial}{\partial u_3} + \cdots + a_{n+1,2} \frac{\partial}{\partial u_n}$$

and $[\triangle_2 \triangle_3] = a_{3,22} \dfrac{\partial}{\partial u_2} + a_{4,22} \dfrac{\partial}{\partial u_3} + \cdots + a_{n+1,22} \dfrac{\partial}{\partial u_n}.$

If it is supposed that $[\triangle_2 \triangle_3]$ is linearly dependent only on $\triangle_2$,

$$a_{3,22} = 0, \ a_{4,22} = 0, \ \ldots, \ a_{n,22} = 0,$$

so that

$$a_i = \phi_i^3 (u_1, \ldots, u_{n-1}) u_n + \psi_i^3 (u_1, \ldots, u_{n-1}), \ 3 \leqslant i \leqslant n$$

and $\triangle_3 = \phi_3^3 \dfrac{\partial}{\partial u_2} + \cdots + \phi_n^3 \dfrac{\partial}{\partial u_{n-1}} + a_{n+1,1} \dfrac{\partial}{\partial u_n}.$

Next apply a transformation I,

I. $v_1 = u_1, v_2 = f_2(u_1, \ldots, u_{n-1}), \ldots, v_{n-2} = f_{n-2}(u_1, \ldots, u_{n-1})$
$v_{n-1} = u_{n-1}, \ v_n = u_n$ where $f_2, \ldots, f_{n-2}$ are $n-3$ inde-

pendent solutions of $\phi_3 \frac{\partial f}{\partial u_2} + \ldots + \phi_n^3 \frac{\partial f}{\partial u_{n-1}} = 0$.

Then $\Delta_3 \equiv \phi_n^3(v_1, \ldots, v_{n-1}) \frac{\partial}{\partial v_{n-1}} + a_{n+1,2}(v_1, \ldots, v_n) \frac{\partial}{\partial v_n}$.

Also $$\Delta_2 \equiv \frac{\partial}{\partial v_n}$$

and $\Delta_3 = \frac{\partial}{\partial v_1} + c_3 \frac{\partial}{\partial v_2} + \ldots + c_{n-1} \frac{\partial}{\partial v_{n-2}} + a_n \frac{\partial}{\partial v_{n-1}} + a_{n+1} \frac{\partial}{\partial v_n}$,

$c_3, \ldots, c_{n-1}$ being functions of $v_1, \ldots, v_{n-1}$ only.

Again apply another transformation II,

II. $w_1 = v_1,\ w_2 = v_2,\ \ldots,\ w_{n-1} = v_{n-1}$,
$w_n = \phi_n^3(v_1, \ldots, v_{n-1})\, v_n + \psi_n^3(v_1, \ldots, v_{n-1})$,

where $\phi_n^3(v_1, \ldots, v_{n-1})$ and $\psi_n^3(v_1, \ldots, v_{n-1})$ are the functions obtained from ϕ_n^3 and ψ_n^3 by the transformation I. Hence as $v_n = u_n$ in the transformation I

$$w_n = \phi_n^3(u_1, \ldots, u_{n-1})\, u_n + \psi_n^3(u_1, \ldots, u_{n-1}) = a_n.$$

Then by II

$$\Delta_3 \equiv \Delta_3 w_{n-1} \frac{\partial}{\partial w_{n-1}} + \Delta_3 w_n \frac{\partial}{\partial w_n}$$

$$= \phi_n^3(w_1, \ldots, w_{n-1}) \left\{ \frac{\partial}{\partial w_{n-1}} + b_{n+1} \frac{\partial}{\partial w_n} \right\},$$

where

$$b_{n+1} = \frac{\partial \phi_n^3}{\partial v_{n-1}} v_n + \frac{\partial \psi_n^3}{\partial v_{n-1}} + a_{n+1,2} = \frac{\partial a_n}{\partial v_{n-1}} + a_{n+1,2}.$$

As it is only necessary to take a differentiator except for a common factor,

$$\Delta_3 = \frac{\partial}{\partial w_{n-1}} + b_{n+1} \frac{\partial}{\partial w_n}.$$

This transforms Δ_2 into $\Delta_2 w_n \frac{\partial}{\partial w_n} = \phi_n^3 \frac{\partial}{\partial w_n}$ and therefore $\Delta_2 = \frac{\partial}{\partial w_n}$. Also Δ_1 transforms to

$$\Delta_1 = \frac{\partial}{\partial w_1} + c_3 \frac{\partial}{\partial w_2} + \dots + c_{n-1} \frac{\partial}{\partial w_{n-2}} + w_n \frac{\partial}{\partial w_{n-1}} + c_{n+1} \frac{\partial}{\partial w_1}$$

where $c_3, \dots, c_{n-1}$ are functions of $w_1, \dots, w_{n-1}$.

Hence it is seen finally that if Δ_1, Δ_2 possess the property that $[\Delta_2 \Delta_3]$ is dependent only on Δ_2, then by suitable topological transformations the two differentiators can be taken, after rejecting common factors, as

$$\Delta_1 = \frac{\partial}{\partial u_1} + a_3 \frac{\partial}{\partial u_2} + \dots + a_{n-1} \frac{\partial}{\partial u_{n-2}} + u_n \frac{\partial}{\partial u_{n-1}} + a_{n+1} \frac{\partial}{\partial u_n}$$

$$\Delta_2 = \frac{\partial}{\partial u_n}, \quad \Delta_3 = \frac{\partial}{\partial u_{n-1}} + b_{n+1} \frac{\partial}{\partial u_n},$$

where $a_3, \dots, a_{n-1}$ are functions of $u_1, u_2, \dots, u_{n-1}$ and both a_{n+1}, b_{n+1} are functions of $u_1, \dots, u_{n-1}, u_n$. Conversely if $\Delta_1, \Delta_2, \Delta_3$ are given by the above expressions, then $a_{i,2} = 0$, $i \leqslant n-1$ and $a_{n,2} = 1$ so that a_i for $i \leqslant n-1$ are functions of $u_1, u_2, \dots, u_{n-1}$ and $a_n = u_n$.

3. Next let us take $\Delta_4 = [\Delta_3, \Delta_1]$, where $\Delta_1, \Delta_2, \Delta_3$ are given by the above forms. Then

$$\Delta_4 = a_{3,3} \frac{\partial}{\partial u_2} + a_{4,3} \frac{\partial}{\partial u_3} + \dots + a_{n-1,3} \frac{\partial}{\partial u_{n-2}} + b_{n+1} \frac{\partial}{\partial u_{n-1}} + (a_{n+1,3} - a_{n+1,23}) \frac{\partial}{\partial u_n}.$$

Hence $$[\Delta_3 \Delta_4] = a_{3,33} \frac{\partial}{\partial u_2} + \dots + a_{n-1,33} \frac{\partial}{\partial u_{n-2}} + b_{n+1,3} \frac{\partial}{\partial u_{n-1}} + (a_{n+1,33} - a_{n+1,233} - a_{n+1,4}) \frac{\partial}{\partial u_n}.$$

Let this depend only on Δ_2 and Δ_3. Then

$$a_{3,33} = 0, \dots, a_{n-1,33} = 0.$$

But $a_{i,3} = \Delta_3 a_i = \frac{\partial a_i}{\partial u_{n-1}} + a_{n+1,2} \frac{\partial a_i}{\partial u_n}$, $3 \leqslant i \leqslant n-1$

$= \frac{\partial a_i}{\partial u_{n-1}}$ for $\frac{\partial a_i}{\partial u_n} = 0$, a_i being functions of $u_1, \dots, u_{n-1}$;

$$a_{i,33} = \frac{\partial^2 a_i}{\partial u_{n-1}^2}.$$

Thus $\frac{\partial^2 a_i}{\partial u_{n-1}^2}=0$, and $a_i=\phi_i^4(u_1,\ldots,u_{n-2})u_{n-1}+\psi_i^4(u_1,\ldots,u_{n-1})$, $3\leqslant i\leqslant n-1$. Repeating the above procedure we get

$$\Delta_1=\frac{\partial}{\partial u_1}+a_3\frac{\partial}{\partial u_2}+\ldots+a_{n-2}\frac{\partial}{\partial u_{n-3}}+u_{n-1}\frac{\partial}{\partial u_{n-2}}$$
$$+u_n\frac{\partial}{\partial u_{n-1}}+a_{n+1}\frac{\partial}{\partial u_n}$$

$$\Delta_2=\frac{\partial}{\partial u_n},\ \Delta_3=\frac{\partial}{\partial u_{n-1}}+c_{n+1}\frac{\partial}{\partial u_n}\quad \Delta_4=\frac{\partial}{\partial u_{n-2}}+c_n\frac{\partial}{\partial u_{n-1}}+d_{n+1}\frac{\partial}{\partial u_n}.$$

Repeating this procedure $(n-3)$ times, we get

$$\Delta_1=\frac{\partial}{\partial u_1}+a_3\frac{\partial}{\partial u_2}+u_4\frac{\partial}{\partial u_3}+\ldots+u_n\frac{\partial}{\partial u_{n-1}}+a_{n+1}\frac{\partial}{\partial u_n}$$

$$\Delta_2=\frac{\partial}{\partial u_n},\ \Delta_3=\frac{\partial}{\partial u_{n-1}}+k_{n+1}^1\frac{\partial}{\partial u_n},\Delta_4=\frac{\partial}{\partial u_{n-2}}+k_n^2\frac{\partial}{\partial u_{n-1}}+k_{n+1}^2\frac{\partial}{\partial u_n}$$

$$\Delta_{n-1}=\frac{\partial}{\partial u_3}+k_5^{n-2}\frac{\partial}{\partial u_4}+\ldots+k_{n+1}^{n-2}\frac{\partial}{\partial u_n},$$

where a_3 is a function of u_1, u_2, u_3 only and k's are functions of u_1, $u_2,\ldots,u_n$.

Now take $\Delta_n=[\Delta_{n-1},\Delta_1]$. Then

$$\Delta_n=(\Delta_{n-1}a_3)\frac{\partial}{\partial u_2}+\ldots=k_3^{n-1}\frac{\partial}{\partial u_2}+\ldots+k_{n+1}^{n-1}\frac{\partial}{\partial u_n}$$

so that $k_3^{n-1}=\Delta^{n-1}a_3=\frac{\partial a_3}{\partial u_3}$ as a_3 is a function of u_1, u_2, u_3 only.

If $\Delta_1,\ldots,\Delta_n$ are linearly independent, then

$$\begin{vmatrix} 1 & a_3 & u_4 & u_5,\ldots,0,0, & u_n, & a_{n+1}\\ 0 & 0 & 0 & .\quad .0,0 & 0 & 1\\ 0 & 0 & 0 & 0\ 0 & 1 & k_{n+1}^1\\ 0 & 0 & 0 & 0\ 1 & k_n^2 & k_{n+1}^2\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 1 & k_{n-2}^{n-2}k_{n-1}^{n-2} & k_n^{n-2} & k_{n+1}^{n-2}\\ 0 & k_3^{n-1} & k_4^{n-1} & \cdots & \cdots & k_{n+1}^{n-1}\end{vmatrix}\neq 0$$

i.e. $$k_3^{n-1} = \frac{\partial a_3}{\partial u_3} \neq 0.$$

Therefore we can take $a_3 = u_3$ and get

$$\Delta_1 = \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \ldots + u_n \frac{\partial}{\partial u_{n-1}} + a_{n+1} \frac{\partial}{\partial u_n}, \text{ and } \Delta_2 = \frac{\partial}{\partial u_n}.$$

The above may be stated in the following form:

If Δ_1 *and* Δ_2 *are two linear differentiators such that* Δ_1, Δ_2, ..., Δ_n *where* $\Delta_n = [\Delta_{n-1}\Delta_1]$ *are* n *independent differentiators and* Δ_2, $\Delta_3, \ldots, \Delta_{n-1}$ *are such that* $[\Delta_{i+1}\Delta_i]$ *is dependent only* Δ_2, $\Delta_3, \ldots, \Delta_i$ *then, by suitable topological transformations, the two differentiators can be brought to the form*

$$\Delta_1 = \frac{\partial}{\partial u_1} + \ldots + u_n \frac{\partial}{\partial u_{n-1}} + a_{n+1} \frac{\partial}{\partial u_n} \text{ and } \Delta_2 = \frac{\partial}{\partial u_n}.$$

The path curves of the differentiator Δ_1 are given by the system of differential equations

$$\frac{du_1}{1} = \frac{du_2}{u_3} = \ldots = \frac{du_{n-1}}{du_n} = \frac{du_n}{a_{n+1}}.$$

It follows that

$$\frac{du_2}{du_1} = u_3, \ \frac{d^2u_2}{du_1^2} = u_4, \ldots, \frac{d^{n-2}u_2}{du_1^{n-2}} = u_n,$$

$$\frac{d^{n-1}u_2}{du_1^{n-1}} = f\left(u_1, u_2, \frac{du_2}{du_1}, \ldots, \frac{d^{n-2}u_2}{du_1^{n-2}}\right).$$

Hence the two differentiators can be associated with a differential equation

$$\frac{d^{n-1}u_2}{du_1^{n-1}} = f\left(u_1, u_2, \frac{du_2}{du_1}, \ldots, \frac{d^{n-2}u_2}{du_1^{n-2}}\right)$$

on the (u_1, u_2) surface.

4. The nature of this association of 2-webs in R_n with an $(n-1)$ parametric system of curves on a surface has the character of a "projection." The lines of the second pencil are the intersections of $(n-1)$ hypersurfaces $u_1 = \text{const}, \ldots, u_{n-1} = \text{const}$. Now consider any arbitrary surface intersecting all lines of the second pencil and con-

taining none of them. This surface is given by $n-2$ equations $\psi_1 = \text{const},\ldots, \psi_{n-2} = \text{const}$. Its intersection with the hypersurface $u_1 = \text{const.}$ is a curve $u_1 = \text{const.}$ on this surface. Similarly its intersection with the hypersurface $u_2 = \text{const}$ is another curve $u_2 = \text{const.}$ on this surface. These two pencils of curves form a parametric net on the surface. The intersection of a particular curve of the second pencil with this surface is a unique point given by $u_1 = \text{const.}$, $u_2 = \text{const.}$ on the surface. This surface will be taken as the base for projection.

Now consider a particular curve (c) of the first pencil. Through each point of (c) there passes only one curve of the second pencil which intersects the base surface in only one point. These curves of the second pencil passing through all points of (c) will generate a surface which may be considered as a projecting surface. The intersection of this and the base surface is a curve (c') on the base surface which may be considered as the projection of (c) by means of the second pencil.

Lastly let us find out the geometrical significance of the condition involved. We know that when brought to to the canonical form,

$$\Delta_2 = \frac{\partial}{\partial u_n}, \quad \Delta_3 = \frac{\partial}{\partial u_{n-1}} + k^1_{n+1}\frac{\partial}{\partial u_n}, \quad \Delta_4 = \frac{\partial}{\partial u_{n-2}}$$

$$+ k^2_n \frac{\partial}{\partial u_{n-1}} + k^2_{n+1}\frac{\partial}{\partial u_n}, \ldots \Delta_{n-1} = \frac{\partial}{\partial u_3} + \ldots + k^{n+1}_{n+1}\frac{\partial}{\partial u_n},$$

$$\Delta_n = \frac{\partial}{\partial u_2} + \ldots + k^n_{n+1}\frac{\partial}{\partial u_n}$$

But $u_1 = \text{const.}\ldots, u_{n-i+1} = \text{const.}$ are among the solutions of $\Delta_i f$

$$= \frac{\partial f}{\partial u_{n-i+2}} + k^i_{n-i+2}\frac{\partial f}{\partial u_{n-i+3}} + k^i_{n-i+3}\frac{\partial f}{\partial u_{n-i+4}} + \ldots + k^i_{n+1}\frac{\partial f}{\partial u_n} = 0.$$

If the $i-1$ dimensional surface which is the intersection of the hypersurfaces $u_1 = \text{const.},\ldots, u_{n-i+1} = \text{const.}$,

be denoted by H_{i-1}, then the path pencil of Δ_i lies on H_{i-1}. Also $H_1 \subset H_2 \subset H_3 \ldots \subset H_{n-2} \subset H_{n-1}$.

The path pencils of $\Delta_2, \Delta_3, \ldots, \Delta_i$ all lie on H_{i-1}.

In order that the system of q dimensional surfaces H_q, $u_1 = \text{const.}, \ldots, u_{n-q} = \text{const.}$ may be invariant for a differentiator

$$Y = p_1(u_1, \ldots, u_n) \frac{\partial}{\partial u_1} + p_2(u_1, \ldots, u_n) \frac{\partial}{\partial u_2} + \ldots + p_n(u_1, \ldots, u_n) \frac{\partial}{\partial u_n}$$

it is necessary and sufficient that $Yu_\nu = p_\nu$, $1 \leqslant \nu \leqslant n-q$ are functions of $u_1, u_2, \ldots, u_{n-q}$ only. Now let the surface H_q be spanned upon by the path pencils of the differentiators $X_1, X_2, \ldots, X_q$. Then

$$X_k = b^k_{n-q+1} \frac{\partial}{\partial u_{n-q+1}} + \ldots + b^k_n \frac{\partial}{\partial u_n}.$$

Therefore $[Y, X_k] = c^k_{n-q+1} \frac{\partial}{\partial u_{n-q+1}} + \ldots + c^k_n \frac{\partial}{\partial u_n}$ and so $[Y, X_x]$ is linearly dependent on $X_1, \ldots, X_q$. This condition is necessary and sufficient in order that the system H_q spanned upon by the path pencils of $X_1, \ldots, X_q$ may be an invariant for Y. Thus it is seen that the path pencils of $\Delta_2, \ldots, \Delta_i$ are spanned over the system of surfaces H_{i-1} which are themselves invariant for Δ_{i+1}.

NOTES ON GROUP THEORY III

HOMOGENEOUS REPRESENTATION OF GROUPS.

BY

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[Received 20 June, 1944.]

In general, groups are determined by the help of an operation, called "composition" by which to every ordered pair a, b of elements a third element $ab = c$ is

attached. There is however a second mode of representing group theory, where instead of an *operation* operating on a pair of elements, a 3-term *relation* is introduced. *E.g.* the equation $abc = 1$ can be replaced by the statement that the relation $R(a, b, c)$ holds. This form of representation leads to the abstract theory of *webs*, which affords a representation of group theory and its generalisations in the form of some combinatorial geometry. By replacing the operational form of investigation by that of "relation", some superficial similarity between the theory of groups and the theory of partially ordered sets is created. The latter theory is based on the two-term relation $a<b$ as much as group theory on the three-term relation $R(a, b, c)$. The difference between the two theories however does not so much result from the number of terms in the fundamental relation as from the fact that in the fundamental relation $R(a, b, c)$ of group theory the element c is uniquely determined by the other terms whereas by $a<b$ for a given a, the element b is in general not uniquely determined. Whenever in a relation one term is uniquely determined by the remaining terms, it is possible to replace the use of relations by operations; the two forms of representation are logically equivalent in these cases. In general it is more convenient to use operations but—as shown by the abstract theory of webs—the representation by relations also deserves some interest.

In this note, the theory of groups will be built up by the help of sets called *pencils* in which a *four-term relation* is given. To each such pencil there corresponds a group which is uniquely determined in the sense of isomorphism and one can map the elements of the pencil on those of the group one by one. However, this $(1,1)$ correspondence is not uniquely determined; it will be shown that besides automorphisms of the group, multiplications with a constant element are admissible; in

other words, one can select an arbitrary pencil-element to be the unit-element of the group. The subpencils correspond to the subgroups and their cosets; the distinction between a subgroup and its cosets comes in only by the selection of an element to serve as unit-element. Thus the pencil is a homogeneous representation of a group which is specially appropriate to such applications of group theory where the distinction between subgroups and their cosets is irrelevant.

The four-term relation corresponds to an operation by which an element is a function of three variables. H. Prufer * has already realised that when a, b, c are three elements of a coset, the element $ab^{-1}c$ belongs to the same coset. Starting from this remark he has established a system of axioms which an abstract three-term operation $ab^{-1}c$ must satisfy in order to get just the equivalent of the subcosets of an abelian group. R. Bær † has solved the same problem in the realm of non-commutative groups; however he did not consider his axioms as a convenient tool for investigation on pencils and he preferred pencils to be considered as subsets of groups which are closed for the operation $ab^{-1}c$. It seems that the representation by relations is more convenient for this purpose as it affords a very simple form of axioms for the pencils and it leads to a combinatorial geometry of that portion of group theory. Moreover the pencils furnish a suitable scheme for a purely group-theoretical foundation of geometry.

1. A set Σ of elements is said to form a *pencil* if and only if a relation

$$\begin{matrix} & b & \\ a & & c \\ & d & \end{matrix} \qquad (1)$$

* *Math. Zeit* 20, 165-87. p. 170. The German term "Schar" is translated in the present paper by "pencil".

† *Jour. f.d. reine und angew. Math.* 160, 199-207 p. 202, footnote.

is defined in Σ which satisfies the following axioms.

AXIOM 1. *Given three elements a, b, c of Σ, then there exists one and only one element d in Σ such that* (1) *holds.*

AXIOM 2. *For every pair of elements a, b of Σ,*

$$(\alpha)\quad \begin{array}{ccc} & a & \\ a & & b \\ & b & \end{array} \qquad \textit{and } (\beta)\quad \begin{array}{ccc} & b & \\ a & & b \\ & a & \end{array} \quad \textit{hold.} \qquad (2)$$

AXIOM 3. *If* (1) *holds, then*

$$(a)\quad \begin{array}{ccc} & f & \\ b & & g \\ & c & \end{array} \qquad \textit{implies} \qquad \begin{array}{ccc} & a & \\ f & & d \\ & g & \end{array} \qquad (3)$$

$$(b)\quad \begin{array}{ccc} & c & \\ d & & f \\ & g & \end{array} \qquad \textit{implies} \qquad \begin{array}{ccc} & f & \\ g & & b \\ & a & \end{array}$$

Put in $(3a): f = b,\ g = c$, and in $(3b): f = c,\ g = d$, then it follows from (2) and (3) that

$$(1)\ \text{implies} \qquad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & c & \\ d & & b \\ & a & \end{array}$$

Thus (1) admits the permutations $(a, b)\ (c, d)$ and $(a, d)\ (b, c)$ and therefore also $(a, c)\ (b, d)$. Hence

THEOREM 1. *The relation* (1) *admits all the permutations of Klein's group of order* 4 "(Vierergruppe").

In geometrical notation: the 'square' (1) admits the symmetries on its medians. From Axiom 1 and Theorem 1 follows immediately

THEOREM 2. *In* (1) *each element is uniquely determined by the other three which can be selected arbitrarily from the elements of* Σ.

2. Select now an arbitrary element e and *define* the product

$$x = ab \qquad (4)$$

of elements a and b of Σ, as the element for which

$\begin{array}{ccc} & e & \\ a & & b \\ & x & \end{array}$ holds. Then $\begin{array}{ccc} & e & \\ e & & b \\ & eb & \end{array}$. From (2) and axiom 1

it follows that $eb = b$ for every b. Moreover there exists [by Theorem 2] to every b an element b^{-1}, such that $b^{-1} \overset{e}{\underset{e}{\quad}} b$ holds and therefore $b^{-1}b = e$. Hence there exists a left hand unity in Σ and a left hand inverse for every element b of Σ. It will be proved now that the associative law holds and that therefore the pencil Σ is a group for the composition determined by e.

By Theorem 1 and axioms (3*a*) and (3*b*), we obtain $ab \overset{b}{\underset{(ab)c}{\quad}} bc$ from $b \overset{e}{\underset{bc}{\quad}} c$ and $ab \overset{e}{\underset{(ab)c}{\quad}} c$ and $ab \overset{b}{\underset{a(bc)}{\quad}} bc$ from $a \overset{e}{\underset{ab}{\quad}} b$ and $a \overset{e}{\underset{a(bc)}{\quad}} bc$. Hence by Ax. 1 $(ab)c = a(bc)$. Therefore, we have

THEOREM 3. *The pencil Σ is a group Σ_e if composition is determined by* (4).

From $b \overset{e}{\underset{bx}{\quad}} x$, $a \overset{e}{\underset{ax}{\quad}} x$ follows $a \overset{b}{\underset{ax}{\quad}} bx$ for arbitrary a, b, x, or for arbitrary $a, b, bx = c$. Hence $a \overset{b}{\underset{ab^{-1}c}{\quad}} c$ holds for every a, b, c. As by (1), the element d is uniquely determined for given a, b, c, we have

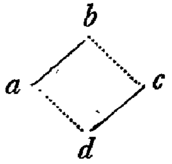
THEOREM 4. *The relation* (1) *holds in Σ if and only if $ab^{-1}cd^{-1} = e$ holds in Σ_e. Therefore $a\,b^{-1}c\,d^{-1}$ is equal to the unit-element of every group Σ_e in whatever manner e may be selected. Independent of the choice of e,*

$$ab^{-1} = dc^{-1} \text{ aud therefore } d^{-1}a = c^{-1}b \tag{5}$$

if and only if (1) *holds in Σ.*

3. The axioms of §1 can also be interpreted in a topological manner. Consider (1) as a quadrangular topological cell with the vertices a, b, c, d and one pair

of opposite "blue" (in print: "dashed") sides and one pair of opposite red (in print: "dotted") sides.



(1′)

Then Σ is made a combinatorial topological complex $C(\Sigma) = C$; its points are the elements of Σ; every pair of points (equal or different) is interconnected by one blue and one red segment and to any two segments of different colours ending at the same point, there exists one and only one neighbouring cell of C. Two directed segments of C will be considered to be *equivalent* if and only if they are opposite and equally directed sides of a cell of C. That this definition of equivalence is admissible, follows from Axioms 2 and 3. In fact these axioms mean that the laws of equivalence are satisfied. (5) shows that to equivalent blue segments, say $(a, b) = (d, c)$ of C there correspond equal elements $\kappa = ab^{-1} = dc^{-1}$ of Σ_e and to equivalent segments $(d, a) = (c, b)$, equal elements $\lambda = d^{-1}a = c^{-1}b$. To every point of C one and only one segment is attached which is equivalent to any given one. Given any blue segment, say $(a, b) = (d, c) = \ldots$; the (1,1) mapping of C on itself which one gets by replacing $b \to a$, $c \to d, \ldots$ can be obtained by multiplying the corresponding elements of Σ_e with κ from the left side. Hence the group of the displacements of C along the blue segments is isomorphic to the group of the left hand similarities of Σ_e and therefore to Σ_e itself. As the group of those displacements does not depend on the selection of e it follows that the groups $\Sigma_a, \Sigma_b, \ldots$ are all isomorphic. Similarly the displacements along a red segment, say $d \to a, c \to b, \ldots$ correspond to the multiplication from the right side by λ and the group of these displacements is isomorphic to Σ_e. Hence,

THEOREM 5. *To every pencil Σ there corresponds a group $G(\Sigma)$ which is uniquely determined in the sense of isomorphism. G is isomorphic to the group of displacements of $C(\Sigma)$ along the blue segments, to the group of displacements of $C(\Sigma)$ along the red segments and to every group Σ_e which we obtain by (4) after selecting an arbitrary element e of Σ as the unit element.*

THEOREM 6. *To every group G there corresponds a uniquely determined pencil Σ formed by the elements of G. Four elements a, b, c, d have in Σ the relation (1) if and only if $ab^{-1}cd^{-1} = 1$ holds in G. Then $G(\Sigma) = G$.*

Let G be a group with the unit element 1 and let composition and inverse element be denoted in the usual manner. Let there be another law of composition denoted by $a \circ b$; for this composition the unit element may be e and the inverse of any element a may be denoted by a'. Thus $a' \circ a = e$. The group formed by the elements of G with this law of composition will be called G'. Then the following theorem holds.

THEOREM 7. *If $a \circ b' \circ c \circ d' = e$ holds if and only if $ab^{-1}cd^{-} = 1$, then*

$$f \circ g = fe^{-1}g \tag{6}$$

for every pair of elements f and g and $G' \simeq G$.

PROOF. G and G' belong to the same pencil Σ and are therefore isomorphic, $f \circ e' \circ g \circ d' = e$ and therefore $f \circ g = d$ holds, if and only if $fe^{-1}g = d$.

On the other hand for every element e of G, we can associate a multiplication (6) and for this multiplication $a \circ b' \circ c \circ d' = e$ if and only if $ab^{-1}cd^{-1} = 1$. Hence

THEOREM 7. *Those and only those properties of G are determined by the pencil Σ which are invariant for a change of the law of composition $fg \rightarrow f \circ g = fe^{-1}g$ for every element e.*

From the law of composition (6) it follows immediately that the inverse element a' can be expressed by

$$a' = e\,a^{-1}e. \tag{6'}$$

Let ϕ be any monomial $a_1^{m_1} \ldots a_t^{m_t}$ in elements $a_1, \ldots, a_t$ of G which are $\neq 1$ and $a_j \neq a_{j+1}$, $a_t \neq a_1$; let ϕ_0 be the corresponding monomial for the composition $\circ$; then:

THEOREM 8. *$\phi = 1$ implies $\phi_0 = e$ in every group and independent of the selection of e if and only if t is even and the exponents $m_1, \ldots, m_t$ take alternately the values $+1$ and -1 (starting with either of them).*

PROOF. If the suppositions of the theorem hold, and we substitute in ϕ_0 for the composition $\circ$ and for the inverse element the values given in (6) and (6′), we obtain $\phi_0 = \phi e = e$ or $e\phi = e$ according as $m_1 = +1$ or $= -1$. If however t is odd and the exponents are alternating $+1, -1$ then we obtain $\phi_0 = \phi = 1$ or $e\phi e = e^2$ according as $m_1 = m_t = +1$ or $= -1$. In all other cases we obtain expressions in $a_1, \ldots, a_t$ and e consisting of the expression ϕ with elements $e^{\pm 1}$ intersown. We see easily that there are less than t consecutive a's in that expression. Suppose now that G is generated by $a_1, \ldots, a_t$ and e with the only generating relation $\phi = 1$, then ϕ is the shortest expression which represents 1 and therefore $\phi_0 \neq e$ unless the suppositions of the theorem are satisfied.

THEOREM 9. *The group of automorphisms of $C(\Sigma)$ is isomorphic to the holomorph of $G(\Sigma)$.*

PROOF. Every automorphism of $C(\Sigma)$ can be composed in one and only one manner, of a displacement along a blue segment and an automorphism which leaves a selected element e invariant. If $e \to e$, $a \to a_1$, $b \to b_1$, then $ab \to a_1 b_1$ the composition being defined according to (4). The automorphisms leaving e invariant form therefore a group which is isomorphic to the group of automorphisms of $\Sigma_e \cong G(\Sigma)$; the displacements along the blue segments form a group isomorphic to $G(\Sigma)$; hence the theorem.

4. Independent of the selection of the unit element e, the blue directed segment leading from b to a corresponds to the group element $a\,b^{-1}$ of Σ_e whereas the red segment corresponds to $b^{-1}a$. Equivalent segments of the same colour correspond to the same group element independent of the selection of the unit element, segments of different colours may correspond to the same or to

different group-elements according to that selection. Suppose that, when e is the unit element, $a\,b^{-1} = \beta^{-1}\alpha$, then $a\,b^{-1}e\alpha^{-1}\beta e^{-1} = 1$, hence $a\,b^{-1}e = e\beta^{-1}\alpha = d$(say). That means: $a\,b^{-1} = de^{-1}$, $\beta^{-1}\alpha = e^{-1}d$. On the other hand if a blue and a red segment are equivalent to segments joining the same points in the same direction, they correspond to the same group-element if one of those points (d or e) is selected as unity. Moreover $a\,b^{-1} = e(\beta^{-1}\alpha)e^{-1}$. Hence:-

THEOREM 10. *If two segments of different colours correspond to conjugate group-elements for one selection of the unit-element, they do so for every selection and they correspond to equal elements for suitable selections. This holds if and only if the two segments are equivalent to segments joining the same couple of points in the same direction.*

This theorem determines completely how far interconnection between segments of Σ can be described by elements of the group $G(\Sigma)$. In what follows, some group theoretical properties of the *points* of Σ will be considered. Suppose e is selected as the unit element; we shall give an invariant form to such relations between points of Σ which depend on that selection and derive therefrom the condition that those relations are satisfied for a suitable selection of the unit-element.

1. Suppose $b = a^{-1}$, then $be^{-1}ae^{-1} = 1$. Hence b and a are opposite vertices of a cell, the other two vertices of which are coincident (namely $= e$).

2. Suppose $b = a^2$, then $ba^{-1}ea^{-1} = 1$. Hence there exists exactly one selection of the unit element such that the condition is satisfied for given a and b.

3. Suppose a and b are commutative, then $ae^{-1}b\;a^{-1}e\,b^{-1} = 1$, hence $ae^{-1}b = be^{-1}a = d$ (say).

Therefore $a\;\overset{e}{\underset{d}{b}}$ and $b\;\overset{e}{\underset{d}{a}}$ hold. i.e. the cell with the vertices a, e, b, d (in cyclic order) admits two different kinds of colouring. Such cells will be called *double cells*.

(E.g. a, a, b, b border a double cell for every a, b). The condition is therefore that a and b *are (opposite) vertices of a double cell in which e occurs as a vertex.* The word "opposite" can be omitted since if a and b are consecutive vertices of the double cell, then a and b^{-1} are commutative and this is equivalent to the commutativity of a and b. If $G(\Sigma)$ is an Abelian group, all cells are double cells and the distinction between red and blue segments is unnecessary.

4. Suppose a_1 and a_2 are conjugate elements, say $c_1a_1c_1^{-1} = a_2$, then $c_1e^{-1}a_1c_1^{-1}e\, a_2^{-1} = 1$, $c_1e^{-1}a_1 = a_2e^{-1}c_1 = d$,

$$\begin{matrix} & d & \\ a_1 & & c_1 \\ & e & \end{matrix} \qquad \begin{matrix} & d & \\ c_1 & & a_2 \\ & e & \end{matrix} .$$

I.e. the points a_1 and a_2 are vertices of two cells which are *neighbouring* along d, c, e (on the common boundary the segments of the neighbouring cells have opposite colours). Define successively $c_2, a_3, c_3 \ldots$ by

$$\begin{matrix} & d & \\ a_i & & c_i \\ & e & \end{matrix} \qquad \begin{matrix} & d & \\ c_i & & a_{i+1} \\ & e & \end{matrix} , \tag{7}$$

then the automorphism $\qquad x \to de^{-1}x\,d^{-1}e \qquad$ (8)

transforms a_i into a_{i+1} and c_i into c_{i+1} whereas d and e remain invariant. The system (7) of cells undergoes a *rotation* about the *axis* d, e. As $a_{i+1} = (de^{-1})^i a_1 (d^{-1}e)^i$, all the a_j—and similarly all the c_j—are conjugate elements when e is selected as the unit element. Thus elements of Σ are *conjugate when they can be transformed into one another by a rotation*, where the unit element is any endpoint of an axis of rotation.

The analogy with the rotations of ordinary geometry goes further. If de^{-1} (and therefore $d^{-1}e$) is of a finite order n, then the cells (7) form a topological sphere of the dihedral type; the number of cells is a factor of $2n$; the rotation (8) generates a geometrical rotation of that dihedron about the axis d, e. Moreover (7) can be considered as the square of the transformation $x \to dx^{-1}e$, which is an "improper" automorphism of Σ interchanging

blue and red segments. By the rotation (8) the axis d, e is not uniquely determined. Sufficient conditions for α, β forming another axis are $d e^{-1} = \alpha \beta^{-1}$, $d^{-1} e = \alpha^{-1} \beta$. This means that both the segments joining d and e are equivalent to the segments of the corresponding colour and orientation which join α and β, in other words, d, e, α, β border a double cell.

5. Let S be a subgroup of $G \simeq \Sigma_e$, then the points of Σ_e which correspond to the elements of S form a set Σ' which is closed for the relation (1), i.e. if 3 vertices of a cell belong to Σ' then the fourth vertex also belongs to it. Σ' will therefore be called a *subpencil* of Σ; it contains the unit element e. By displacing Σ' along equivalent blue segments, we obtain another subpencil corresponding to a coset of S, say to $a b^{-1} S$. As two cosets of S are either identical or have no element in common, we can resolve Σ into a set of subpencils without common points; these subpencils are similar to one another by left side similarity. (*Blue resolution of G by S.*) On the other hand if any subpencil Σ'' of Σ is given, it corresponds to a subgroup of G only if the unit element is selected out of its points, but this condition is also sufficient since from (4) it follows that if a and b belong to Σ, a^{-1} and ab also belong to it. The "blue resolution" is independent of the selection of the unit element. Hence every subpencil corresponds to a subgroup or to a (left side) coset of it according as the unit element lies in it or not. Correspondingly there exists to every subpencil a *red resolution* which is generated by a right side similarity. Every blue (red) resolution is uniquely determined by each of its subpencils, but in general the two resolutions generated by the same subpencil are different. If and only if the subgroup corresponding to one of the subpencils (and therefore to each of them) is a normal subgroup, the two resolutions coincide. From §4.4 is seen that in *a normal resolution* a subpencil Σ' is invariant for rotations when the end point

of an axis lies in Σ'. It is not difficult to describe group properties in the language of the geometry of pencils. E.g. if S is a subgroup of the centre of G, then the corresponding ubpencil Σ' has the property that when two vertices of a cell lie in Σ' (or in any subpencil of its resolution), the cell is a double cell (see §4.3).

The meet of the two subpencils of Σ is a subpencil. Let Σ_1, Σ_2,..be a blue resolution of Σ and Σ^1, Σ^2,...be another one; let a be any point of $\Sigma_i \cap \Sigma^k$ and b a point of $\Sigma_m \cap \Sigma^n$. Then there exists one and only one left side displacement of Σ carrying a into b; by this operation Σ_1, Σ_2,...are interchanged among themselves and similarly $\Sigma^1, \Sigma^2,\ldots$; in particular Σ_i is carried into Σ_m and Σ^k into Σ^n; hence $\Sigma_i \cap \Sigma^k$ into $\Sigma_m \cap \Sigma^n$. The meets of the two blue resolutions therefore form again a blue resolution. The corresponding holds for two red resolutions. However the meet of a blue and a red resolution is a partition of Σ into subpencils which in general is not a resolution. The partitions of Σ into subpencils form a lattice, the generating binary relation being a "refinement" of the partition. The blue and the red resolutions form two sublattices of it; these and their sublattices are of particular interest.

6. In a recent paper* S. Chern has made the suggestion to build up geometry without the use of geometrical elements, in a purely group theoretical way. If G is the group of the given geometry then every point A is represented by the subset of G consisting of the transformations carrying a particular point O into A, this set is a left side coset of the subgroup of G for which O is invariant. Similarly a straight line (plane, conic, etc.) s can be represented by a set of transformations carrying a given *canonical* straight line (plane, conic, etc.) into s. It is supposed that to different geometrical elements belong different sets of transformations.†. If the canonical ele-

* *Annals of Math.* 43, 178-89.

† Otherwise we obtain a geometry which is only homomorphic to the required one e.g. if we apply the method to the metric geometry on the sphere, we obtain elliptic plane geometry.

ments are incident, then obviously two arbitrary elements are incident if and only if the corresponding sets of transformations have common elements. Thus the canonical elements are represented by subgroups, the other elements by left side cosets. This distinction of canonical elements—which is the last remnant of the coordinate system drawn in the space—can be avoided, if instead of the group G the corresponding pencil Σ (where $G(\Sigma) = G$) is used. To every class of geometrical objects (say points, or straight lines, ...) there corresponds a blue resolution of Σ; the subpencils of the resolution represent the single objects. From this point of view* *a geometry is a sublattice L of the lattice of the blue resolutions of a pencil* Σ. Two geometries belonging to different (non-isomorphic) groups may nevertheless be lattice-isomorphic. It appears that the properties in such a geometry consist of two strata, wherein those of the lower stratum concern lattice properties only. Besides the lattice of the resolution we have also to consider the lattice of the subpencils occurring in the geometry. To every subpencil there exists a lattice of which it is the lower limit and one of which it is the upper limit. If A is the upper limit of L_1 and B the upper limit of L_2, then A and B are incident if and only if $L_1 \cap L_2$ is non-empty. Thus, incidence is one of the simplest lattice properties of a geometry. Seen from this angle, the lattice properties of a geometry, seem to form some kind of "geometry of position" whereas the group- (or pencil) elements and the sets of them play rather the role of quantities. If one selects in Σ a particular element e as unit element, the subpencils containing e are made subgroups and therefore in each blue resolution belonging to L one subpencil is distinguished as a canonical element. Thus the selection of the unit element stands here for the selection of a system of coordinates.

* It is the group-theoretical point of view of the "Erlanger Programm." It need hardly be emphasized that no definition embracing all the possibilities for geometries is intended to be given here.

SOME INFINITE INTEGRALS INVOLVING CYLINDER FUNCTIONS.

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[Received 10th March 1944]

1. Since[1] $D_n(ip)\,D_{-m-1}(p)$

$$=i^n n!\,e^{\frac{1}{4}p^2}\sum_{r=0}^{n}\frac{(-1)^r\,\Gamma(m-n+2r+1)}{(n-r)!\,r!\,\Gamma(m-n+r+1)}\,p.D_{-(m-n+2r)-1}(p) \qquad (1.1)$$

we have on interpreting term by term, with the help of

$$\Gamma(\lambda+1)p.D_{-\lambda-1}(p).e^{\frac{1}{4}p^2}\doteqdot e^{-\frac{1}{2}x^2}.x^\lambda,\quad R(\lambda+1)>0 \qquad (1.2)$$

a new operational relation[2]

$$\Gamma(m+1)\,p\,D_n(ip)D_{-m-1}(p)\doteqdot i^n n!\;e^{-\frac{1}{2}x^2}\,x^{m-n}L_n^{m-n}(x^2),\quad R(m-n+1)>0 \qquad (1.3)$$

or

$$\Gamma(2m+n+1)pD_n(ip)D_{-2m-n-1}(p)\doteqdot i^n n!\;e^{-\frac{1}{2}x^2}x^{2m}L_n^{2m}(x^2),\quad R(2m+1)>0. \qquad (1.4)$$

Hence using (1.4) with

$$x^\nu J_\nu(ax)\doteqdot\frac{a^\nu\Gamma(2\nu+1)}{2^\nu\Gamma(\nu+1)}.\;\frac{p}{(a^2+p^2)^{\nu+\frac{1}{2}}} \qquad (1.5)$$

in Parceval's formula we have,

$$\int_0^\infty D_n(it)D_{-n-2m-1}(t)\,t^\nu J_\nu(xt)\,dt$$

$$=\frac{i^n n!\,\Gamma(2\nu+1)x^\nu}{2^\nu\Gamma(n+2m+1)\Gamma(\nu+1)}\int_0^\infty\frac{e^{-\frac{1}{2}t^2}t^{2m}L_n^{2m}(t^2)}{(x^2+t^2)^{\nu+\frac{1}{2}}}\,dt. \qquad (1.6)$$

Expanding $L_n^{2m}(t^2)$ in powers of t and integrating term by term with the aid of the formula[3],

$$\int_0^\infty\frac{e^{-\frac{1}{2}t^2}t^l}{(x^2+t^2)^m}dt=\frac{\Gamma[\frac{1}{2}(l+1)]\;x^{l/2-m-\frac{1}{2}}\,e^{\frac{1}{4}x^2}}{2^{\frac{1}{4}(2m-l+1)}}\,W_{-\frac{1}{4}(2m+l-1),\,\frac{1}{4}(l-2m+1)}(\tfrac{1}{2}x^2) \qquad (1.7)$$

we find that

$$\int_0^\infty D_n(it)D_{-n-2m-1}(t)\,t^{\nu-\frac{1}{2}}\,J_\nu(xt)\,\sqrt{xt}\,dt$$

(1) Hari Shanker: *Proc. Benares Math. Soc.* (2), 2 (1940).

(2) Hari Shanker: *Proc. Nat. Academy of Sciences*, 2 (1941) 73-7.

(3) Hari Shanker: *Jour. Ind. Math. Soc.* (2) 7 (1943), 63-7.

$$= \frac{i^n n!\,\Gamma(2\nu+1)\,2^{\frac{1}{2}(m-3\nu-1)} e^{\frac{1}{4}x^2}\,x^{m-\frac{1}{2}}}{\Gamma(\nu+1)} \sum_{r=0}^{n} \frac{\Gamma(m+r+\frac{1}{2})\,(-\sqrt{2x})^r}{(n-r)!\,r!\,\Gamma(2m+r+1)}$$

$$\times W_{-\frac{1}{2}(m+r+\nu),\,\frac{1}{2}(m+r-\nu)}\left(\tfrac{1}{2}x^2\right),$$

$R(2m+1)>0$, $R(\nu+\frac{1}{2})>0$ and $R(\nu-2m+\frac{1}{2})<0$. (1.8)

Similarly using $x^{\nu+1}J_\nu(ax) \doteqdot \frac{a^\nu\Gamma(2\nu+2)}{2^\nu\Gamma(\nu+1)} \frac{p^2}{(a^2+p^2)^{\nu+\frac{3}{2}}}$ (1.9)

in place of (1.5), we get

$$\int_0^\infty D_n(it)D_{-n-2m-1}(t)\,t^{\nu+\frac{1}{2}}J_\nu(xt)\sqrt{xt}\,dt$$

$$= \frac{i^n n!\,\Gamma(2\nu+2)\,2^{\frac{1}{2}(m-3\nu)-\frac{3}{4}}}{\Gamma(\nu+1)} \cdot e^{\frac{1}{4}x^2}\,x^{m-1}$$

$$\times \sum_{r=0}^{n} \frac{\Gamma(m+r+1)(-\sqrt{2x})^r}{(n-r)!\,r!\,\Gamma(2m+r+1)} W_{-\frac{1}{2}(m+\nu+r)+\frac{3}{4},\,\frac{1}{2}(m-\nu+r)-\frac{1}{4}}\left(\tfrac{1}{2}x^2\right)$$

$R(2m+1)>0$, $R(\nu+1)>0$ and $R(\nu-2m+\frac{1}{2})<0$. (1.10)

These results together with the corresponding reciprocal relations which can be proved with the help of my paper(3) give rise to the following theorems:—

THEOREM 1. *If* $\phi(x) = x^{\nu-\frac{1}{2}}D_n(ix)D_{-n-2m-1}(x)$

and $$\psi(x) = \frac{i^n n!\,\Gamma(2\nu+1)\,2^{\frac{1}{2}(m-3\nu-1)}}{\Gamma(\nu+1)} e^{\frac{1}{4}x^2}x^{m-\frac{1}{2}}$$

$$\times \sum_{r=0}^{n} \frac{\Gamma(m+r+\frac{1}{2})\,(-\sqrt{2x})^r}{(n-r)!\,r!\,\Gamma(2m+r+1)} W_{-\left(\frac{m+\nu+r}{2}\right),\,\left(\frac{m-\nu+r}{2}\right)}\left(\tfrac{1}{2}x^2\right)$$

then $\phi(x)$ *and* $\psi(x)$ *are Hankel-Transforms of each other of order* ν *provided that* $R(2m+1)>0$, $R(\nu+\frac{1}{2})>0$ *and* $R(\nu-2m-\frac{1}{2})<0$, *n being a positive integer or zero.*

THEOREM 2. *If* $\phi(x)=x^{\nu+\frac{1}{2}}D_n(ix)D_{-n-2m-1}(x)$ *and*

$$\psi(x)= \frac{i^n n!\,\Gamma(2\nu+2)}{\Gamma(\nu+1)} \cdot 2^{\frac{m-3\nu}{2}-\frac{3}{4}} \cdot e^{\frac{1}{4}x^2}\,x^{m-1}$$

$$\times \sum_{r=0}^{n} \frac{\Gamma(m+r+1)\,(-\sqrt{2x})^r}{(n-r)!\,r!\,\Gamma(2m+r+1)} W_{-\left(\frac{m+\nu+r}{2}+\frac{3}{4}\right),\,\left(\frac{m-\nu+r}{2}-\frac{1}{4}\right)}\left(\tfrac{1}{2}x^2\right).$$

then $\phi(x)$ *and* $\psi(x)$ *are Hankel-Transforms of each other of order* ν *provided that* $R(2m+1)>0$ *and* $0<R(\nu+\frac{3}{4})<R(2m+\frac{1}{4})$, *n being a positive integer.*

Particular cases have been previously given by R. S. Varma.(4)

Since,

$$\frac{d^m}{dz^m}\left[L_n^{\alpha}(z)\right]=(-1)^m L_{n-m}^{\alpha+m}(z) \text{ for } m\leq n$$

and $$=0 \text{ for } m>n, \qquad (2.1)$$

we have by Taylor's Theorem,

$$L_n^{\alpha}(z+t)=\sum_{r=0}^{n} L_{n-r}^{\alpha+r}(t)\frac{(-z)^r}{r!} \qquad (2.2)$$

Using this expansion and the formula(5)

$$\int_0^{\infty} e^{-\frac{1}{4}(z+t)^2} D_{\mu}(z+t)t^{\lambda-1}\,dt=\Gamma(\lambda)\, e^{-\frac{1}{4}z^2} D_{\mu-\lambda}(z),\ R(\lambda)>0 \qquad (2.3)$$

we have

$$\int_0^{\infty} e^{-\frac{1}{4}(z-t)^2} D_{\mu}(z+t)\, L_n^{\alpha}(ax^2+bt^2)\, t^{\nu-1}dt$$

$$=e^{-\frac{1}{4}z^2}\sum_{r=0}^{n}\frac{\Gamma(\nu+2r)\,D_{\mu-\nu-r}(z)L_{n-r}^{\alpha+r}(ax^2)(-b)^r}{r!},\ R(\nu)>0. \quad (2.4)$$

Special cases of this integral are worth mentioning.

(i) $$\int_0^{\infty} e^{-\frac{1}{4}(z+t)^2}D_{n+m}(z+t)L_n^{m-\frac{1}{2}}(t^2/4)\, t^{2m-1}\,dt$$

$$=e^{-\frac{1}{4}z^2}\sum_{r=0}^{n}\frac{\Gamma(2m+2r)\ \Gamma(n+m+\frac{1}{2})(-1)^r}{2^{2r}\,(n-r)!\,r!\,\Gamma(m+r+\frac{1}{2})}D_{n-m-2r}(z) \quad (2.5)$$

$$=\frac{2^{2m-1}\,\Gamma(n+m+\frac{1}{2})\ \Gamma(m)}{n!\,\sqrt{\pi}}D_n(z)D_{-m}(z)^{(1)},\ R(m)>0. \quad (2.6)$$

(ii) $$\int_0^{\infty} e^{-\frac{1}{2}(z+t)^2}\, t^{m-n-1}L_n^{m-n-1}(t^2)\,dt$$

$$=e^{-\frac{1}{4}z^2}\Gamma(m)\sum_{r=0}^{n}\frac{\Gamma(m-n+2r)\,(-1)^r}{(n-r)!\,r!\,\Gamma(m-n+r)}D_{-(m-n)-2r}(z) \quad (2.7)$$

(4) R. S. Varma: *Proc. Lond. Math. Soc.* (2) 42 (1936), 9-17, and *Jour. Ind. Math. Soc.* (2) (1937), 269-75.

(5) W. T. Howell: *Phil.Mag.* (7) 24(1937) 396-405, 1082-1093.

$$=\frac{\Gamma(m)}{\Gamma(n+1)} i^{-n} e^{-\frac{1}{2}z^2} D_n(iz)D^{-m}(z) \text{ by (1.1), } R(m-n)>0$$

n being a positive integer or zero. (2.8)

$$(iii) \int_0^{\infty} e^{-\frac{1}{4}(z+t)^2} D_{2n+2m} (z+t) L_n^{m-1} (t^2/2) t^{2m-1} dt$$

$$=\Gamma(n+m) e^{-\frac{1}{4}z^2} \sum_{r=0}^{n} \frac{\Gamma(2m+2r)}{(n-r)!\, r!\, 2^r} \frac{(-1)^r}{\Gamma(m+r)} D_{2n-2r}(z),$$

n being integral and $R(m)>0$. (2.9)

Since [6] $$L_n^{\alpha}(y)=\sum_{r=0}^{n} \frac{\Gamma(\alpha-\beta+r)}{\Gamma(\alpha-\beta)r!} L_{n-r}^{\beta}(y) \quad (2.10)$$

and $$L_n^{-\frac{1}{2}}(y^2/2)=\frac{(-1)^n e^{\frac{1}{4}y^2}}{n!\, 2^n} D_{2n}(y) \quad (2.11)$$

we have from (2.10) for $\beta=-\frac{1}{2}$, $\alpha=m$, and $y=x^2/2$

$$(-1)^n L_n^m(x^2/2)=\frac{e^{\frac{1}{4}x^2}}{2^n\, \Gamma(m+\frac{1}{2})} \sum_{r=0}^{n} \frac{\Gamma(m+r+\frac{1}{2})}{(n-r)!\, r!} (-2)^r D_{2n-2r}(x). \quad (2.12)$$

Hence using (2.12) in (2.9), we obtain

$$\int_0^{\infty} e^{-\frac{1}{4}(z+t)^2} D_{2n+2m} (z+t) L_n^{m-1} (t^2/2) t^{2m-1} dt$$

$$=\frac{\Gamma(n+m)\Gamma(m+\frac{1}{2})}{(\pi)^{\frac{1}{2}}} 2^{2m+n-1} (-1)^n e^{-\frac{1}{2}z^2} L_n^m(z^2/2), R(m)>0. \quad (2.13)$$

Hence again for $m=\frac{1}{2}$, we get

$$\int_0^{\infty} e^{-\frac{zt}{2}} D_{2n}^{\frac{1}{2}}(t)\, D_{2n+1}(z+t)\, dt =\frac{(-1)^n\, \Gamma(n+\frac{1}{2})\, 2^n\, D_{2n+1}(z)}{\sqrt{\pi z}}. \quad (2.14)$$

(6) Hari Shanker: *Jour. Ind. Math. Soc.* (2) 1940, 34-8.

ERRATA

S. Minakshisundaram: On Expansion in Eigenfunctions of Boundary Value Problems V. J.I.M.S. 7 (1943), 89-95.

Delete §§3.1, 3.2 and the second para in §5.2.

HIGHLY COMPOSITE NUMBERS OF THE tTH ORDER.*

BY

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[Received 23 November 1943].

1. Let $d_t(N)$ denote the number of ways of decomposing N into t factors. N is defined as a *highly composite number of the t th order*, if $d_t(N) > d_t(N')$ for all $N' < N$. Ramanujan's highly composite numbers form the particular case when $t = 2$. Closely following Ramanujan's paper, all his results can be generalized. It is the object of this paper to record these generalizations.

λ, μ, ν, p denote primes. All numbers are positive integers unless otherwise stated; $[x]$ is the integral part of x. P is the prime next above p. N stands for a typical highly composite number of the t th order, which will be called "highly composite", whenever no confusion is likely.

If
$$N = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r},$$
then
$$d_t(N) = \prod_{s=1}^{r} \frac{(a_s+1)\ (a_s+2)\ldots(a_s+t-1)}{(t-1)!}$$
Since
$$d_t\ (\pi_2^{a_2}\pi_3^{a_3} \ldots \pi_p^{a_p}) = d_t(2^{a_2}3^{a_3}\ldots p^{a_p}),$$
where $\pi_2, \pi_3, \ldots, \pi_p$ are any primes, it follows that
$$N = 2^{a_2}3^{a_3}\ldots p^{a_p},\ a_2 \geqslant a_3 \geqslant \ldots \geqslant a_p \geqslant 1.$$

2. Let $N' = N\lambda^n/\nu$, where $n = [\log \nu/\log \lambda]$, $\nu \leqslant p$. N' is evidently less than N, and hence
$$\frac{(a_\nu+1)\ldots(a_\nu+t-1)}{(t-1)!} \cdot \frac{(a_\lambda+1)\ldots(a_\lambda+t-1)}{(t-1)!}$$
$$> \frac{a_\nu(a_\nu+1)\ldots(a_\nu+t-2)}{(t-1)!} \cdot \frac{(a_\lambda+n+1)\ldots(a_\lambda+n+t-1)}{(t-1)!}$$

* Part of the thesis for the D.Sc. of the Madras University.

Hence,

$$1+\frac{t-1}{a_\nu} > \left(1+\frac{n}{a_\lambda+1}\right)\cdots\left(1+\frac{n}{a_\lambda+t-1}\right)$$

$$\geqslant \left(1+\frac{n}{a_\lambda+t-1}\right)^{t-1} \geqslant 1+\frac{n(t-1)}{a_\lambda+t-1}.$$

Hence, $a_\lambda+t-1 > na_\nu$.

So, $a_\lambda \geqslant na_\nu - t+2$, since a_λ is an integer.

Thus, $a_\lambda \geqslant a_\nu\,[\log \nu/\log \lambda]-t+2.$ (1)

Next take $N' = NP/2p$. Since $d_t(P) = t$, and $N' < N$,

$$(a_2+1)\ldots(a_2+t-1).(a_p+1)\ldots(a_p+t-1)$$
$$> a_2\ldots(a_2+t-2).a_p(a_p+1)\ldots(a_p+t-2)\,t.$$

Hence $a_2 < (a_p+t-1)/(a_p-1).$ (2)

Since $a_p \leqslant a_2$, this gives $a_p^2-a_p < a_p+t-1$, or

$a_p < 1+t^{\frac{1}{2}}$, when $p \geqslant 3$. (3)

Further, if $a_2 \geqslant (r+t-1)/(r-1)$ we have from (2), $a_p \leqslant r-1$.

Putting $r = 2$, we get $a_p = 1$ when $a_2 \geqslant t+1$.

If $p \geqslant 2^t$, and $a_p \geqslant 2$, we get from (1) by putting $\lambda = 2$ and $\nu = p_2$ that $a_2 \geqslant t+2$, and hence $a_p = 1$. Thus, $a_p = 1$ when $N > N_0(t)$. So we have $a_p = 1$ *for all except a finite number of highly composite numbers. In no case can* $a_p > 1+t^{\frac{1}{2}}$ *except when* $p = 2$.

We shall also show that if $p = 2$, then $a_2 < (t+\frac{1}{4})^{\frac{1}{2}}+\frac{3}{2}$. Let $N = 2^{a_2}$ be highly composite and $a_2 \geqslant 2$. Taking $N' = 3N/4$, we have

$$(a_2+1)\ldots(a_2+t-1) > (a_2-1)\ldots(a_2+t-3)t,$$

i.e. $$(a_2+t-2)\,(a_2+t-1) > ta_2(a_2-1)$$

i.e. $$a_2^2 < 3a_2+t-2,$$

which gives the result. We shall hereafter assume that $N > N_0(t)$, so that $a_p = 1$.

3. Let $m = [\log \mu/\log \lambda] < a_\lambda$, $N' = N\mu\lambda^{-m-1}$. Then $N' < N$ and is an integer.

So, $(a_\mu+1)\ldots(a_\mu+t-1).(a_\lambda+1)\ldots(a_\lambda+t-1)$
$>(a_\mu+2)\ldots(a_\mu+t).(a_\lambda-m)\ldots(a_\lambda-m+t-2).$

Hence,

$$1-\frac{t-1}{a_\mu+t}>\left(1-\frac{m+1}{a_\lambda+1}\right)\ldots\left(1-\frac{m+1}{a_\lambda+t-1}\right)$$
$$>\left(1-\frac{m+1}{a_\lambda+1}\right)^{t-1}\geqslant 1-\frac{(t-1)(m+1)}{a_\lambda+1},$$

for $m+1<a_\lambda+1$ by assumption. Hence

$$a_\lambda+1<(m+1)(a_\mu+t) \text{ when } m \not< a_\lambda.$$

But this is certainly true when $m>a_\lambda$. Hence, using (1), we have when $\nu\leqslant p$, and μ is any prime,

a_ν $[\log\nu/\log\lambda]-t+2\leqslant a_\lambda\leqslant(a_\mu+t)\left\{[\log\mu/\log\lambda]+1\right\}-2,$
where $\nu\leqslant p$. (4)

If we take $\nu=p$, $\mu=P$ in (4), we get $a_\nu=1$, $a_\mu=0$. So,

$$[\log p/\log\lambda]-t+2\leqslant a_\lambda\leqslant t[\log P/\log\lambda]+t-2. \quad (5)$$

4. Let* $m=\sqrt{(k\log\mu/\log\lambda)}$, $n=\sqrt{(k\log\lambda/\log\mu)}$, $N'=N\lambda^{[m]}\mu^{-1-[n]}$, $k=a_\lambda+a_\mu+t-1$.

If we assume $a_\mu>n$, N' will be less than N, and so

$(a_\lambda+1)\ldots(a_\lambda+t-1).(a_\mu+1)\ldots(a_\mu+t-1)$
$(a_\lambda>+[m]+1)\ldots(a_\lambda+[m]+t-1)(a_\mu-[n])\ldots(a_\mu-[n]+t-2).$

So, $\quad(a_\lambda+t-1)(a_\mu+t-1)>(a_\lambda+m)\ (a_\mu-n).$ (6)

The restriction that $a_\mu>n$ is now unnecessary, since (6) is evidently true when $a_\mu\leqslant n$. From (6) we get

$$ma_\mu-na_\lambda<k(t-1)+mn=kt.$$

Hence for all λ and μ, $a_\mu\log\mu-a_\lambda\log\lambda<t\sqrt{(k\log\mu\log\lambda)}$.

Since λ and μ can be interchanged, it follows that

$$|a_\mu\log\mu-a_\lambda\log\lambda|<t\sqrt{(k\log\lambda\log\mu)}. \quad (7)$$

From Bertrand's postulate, $\log P=\log p+O(1)$.

Hence from (5) we get $a_\nu\log\nu\asymp\log p$, when $\nu\leqslant p$, and from (7),

$$a_\mu\log\mu-a_\lambda\log\lambda=O\sqrt{(\log p\log\lambda\mu)}. \quad (7')$$

Hence,

if $\log\lambda=o(\log p)$, *then* $a_2\log 2\sim a_3\log 3\sim\ldots\sim a_\lambda\log\lambda$. (8)

5. Writing $x = a_\lambda \log\lambda$, $y = a_\mu \log\mu$, $m = \log\mu$, $n = \log\lambda$, (7) gives

$$|x - \nu| < t\sqrt{\{(t-1)\log\lambda\log\mu + (a_\lambda + a_\mu)\log\lambda\log\mu\}}$$
$$\lessdot t\sqrt{(tmn + mx + ny)} < t\sqrt{(t^2mn + mx + ny)}.$$

But when this condition is satisfied, we have

$$\left.\begin{array}{l} |\sqrt{(x+t^2n)} - \sqrt{(y+t^2n)}| < t\sqrt{(m+n)} \\ |\sqrt{(x+t^2n)} - t\sqrt{(m+n)}| < \sqrt{(y+t^2n)} \end{array}\right\}. \qquad (9)$$

This follows from formula (62) of Ramanujan's memoir on highly composite numbers, after changing m and n to t^2m and t^2n.* Substituting for x, y, etc. changing μ into ν, from the first equation in (9) transferring and squaring, we get,

$$(t^2+a_\lambda)\log\lambda < (t^2+a_\nu)\log\nu + t^2\log\lambda\nu + 2t\sqrt{\{(t^2+a_\nu)\log\nu\log\lambda\nu\}}. \qquad (10)$$

If $\lambda \leqslant \mu$, we have

$$a_\lambda\log\lambda + (t-1)\log\mu < a_\lambda\log\lambda + t^2\log\mu$$
$$< \{\sqrt{(t^2+a_\nu)\log\nu} + t\sqrt{\log\mu\nu}\}^2, \text{ by (10)}. \qquad (11)$$

Similarly, from the second equation in (9), we obtain

$$a_\lambda\log\lambda + t^2\log\mu > \{\sqrt{(t^2+a_\nu)\log\nu} - t\sqrt{\log\mu\nu}\}^2. \qquad (12)$$

Let $m_\nu = (\log\lambda)/(\pi(\mu)\log\nu)$, $\pi(\mu)\log\mu < \log\lambda \leqslant \log p$,

$$N' = \frac{N}{\lambda}\, 2^{[m_2]}\, 3^{[m_3]} \ldots \mu^{[m_\mu]}.$$

Since $N' < N$, we get*

$$(1+(t-1)/a_\lambda)\,\{(a_2+t-1)\ldots(a_\mu+t-1)\}^{t-1} > \{(a_2+m_2)\ldots \times (a_\mu+m_\mu)\}^{t-1}.$$

So,

$$(1+(t-1)/a_\lambda)\{(a_2\log 2 + \overline{t-1}\log\mu)\ldots(a_\mu\log\mu + \overline{t-1}\log\mu)\}^{t-1}$$
$$> \{(a_2\log 2 + \log\lambda/\pi(\mu)\ldots(a_\mu\log\mu + \log\lambda/\pi(\mu)\}^{t-1}.$$

Hence, we get*, using (11),

$$1 + \frac{t-1}{a_\lambda} > \left\{1 + \frac{\log\lambda/\pi(\mu) - (t-1)\log\mu}{\{\sqrt{(t^2+a_\nu)\log\nu} + t\sqrt{\log\mu\nu}\}^2}\right\}^{(t-1)\pi(\mu)}$$

* See Ramanujan's memoir.

Hence, $\sqrt{(t^2+a_\nu)\log\nu}+t\sqrt{\log\mu}$

$$> \left\{\frac{\log\lambda/\pi(\mu)-(t-1)\log\mu}{(1+(t-1)/a_\lambda)^{1/(t-1)\pi(\mu)}-1}\right\}^{\frac{1}{2}}. \qquad (13)$$

Next take $\quad N'=\lambda N 2^{-1-[m_2]}3^{-1-[m_3]}\ldots\mu^{-1-[m_\mu]}$

$$a_k\log k>\log\lambda/\pi(\mu),\quad k\leqslant\mu<\lambda\leqslant p.$$

We get as before

$$\left(\frac{1+a_\lambda}{t+a_\lambda}\right)\left\{(a_2\log 2+t^2\log\mu)\ldots(a_\mu\log\mu+t^2\log\mu)\right\}^{t-1}$$

$$>\left\{(a_2\log\mu-\log\lambda/\pi(\mu))\ldots(a_\mu\log\mu-\log\lambda/\pi(\mu))\right\}^{t-1}$$

which gives

$$\sqrt{(t^2+a_\nu)\log\nu}-t\sqrt{\log\mu\nu}<\left\{\frac{\log\lambda/\pi(\mu)+t^2\log\mu}{1-\left\{(1+a_\lambda)/(t+a_\lambda)\right\}^{1/(t-1)\pi(\mu)}}\right\}^{\frac{1}{2}}. \qquad (14)$$

But if $\quad a_k\log k\leqslant\log\lambda/\pi(\mu)$, the left side of (14)

$$<\sqrt{(a_\lambda\log\lambda+t^2\log\mu)}<\sqrt{\left\{\log\lambda/\pi(\mu)+t^2\log\mu\right\}}$$

$<$ the right side of (14).

Hence (14) holds good independent of the magnitude of $a_k\log k$.

6. Putting $\nu=2$ in (13), we have

$$(t^2+a_2)\log 2+t^2\log 2\mu+2t\sqrt{\left\{(t^2+a_2)\log 2\log 2\mu\right\}}$$

$$>\frac{\log\lambda/\pi(\mu)-(t-1)\log\mu}{\left\{1+(t-1)/a_\lambda\right\}^{1/(t-1)\pi(\mu)}-1}.$$

Now, since when $x>0$, $(e^x-1)^{-1}=1/x+O(1)$,

$$\frac{\log\mu}{\left\{1+(t-1)/a_\lambda\right\}^{1/(t-1)\pi(\mu)}-1}=O(\mu a_\lambda).$$

Also $\qquad \log\lambda/\pi(\mu)=O(\log\lambda\log\mu/\mu)$

$$\log\lambda\log\mu/\mu+\mu\log p/\log\lambda>2\sqrt{(\log p\log\mu)}.$$

Hence

$$\sqrt{(\log p\log\mu)}=O(\log\lambda\log\mu/\mu+\mu\log p/\log\lambda),$$

$$\mu a_\lambda=O(\mu\log p/\log\lambda).$$

From all these results and (5), we have

$$a_2 \log 2 > \frac{(t-1)\log\lambda}{\log\{1+(t-1)/a_\lambda\}} + O\left\{\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p}{\log\lambda}\right\}. \quad (15)$$

Similarly starting from (14), we obtain

$$a_2 \log 2 < \frac{(t-1)\log\lambda}{\log\{(t+a_\lambda)/(1+a_\lambda)\}} + O\left\{\frac{\log\lambda\log\mu}{\mu} + \frac{\mu\log p}{\log\lambda}\right\}. \quad (16)$$

Put $\lambda = p$ in (15) and $\lambda = P$ in (16) and choose μ so that

$$\mu \asymp \sqrt{(\log p \log\log p)}.$$

Then since $\log P = \log p + O(1)$, we have

$$a_2 \log 2 = \frac{(t-1)\log p}{\log t} + O\sqrt{(\log p \log\log p)}. \quad (16')$$

Hence from (7′), $a_\lambda \log\lambda = (t-1)\log p/\log t$

$$+ O[\sqrt{(lpl\lambda)} + \sqrt{(lpl_2p)}].^* \quad (17)$$

Hence when $\log\lambda = o(\log p)$,

$$a_2 \log 2 \sim \ldots \sim a_\lambda \log\lambda \sim (t-1)\log p/\log t.$$

7. Let $\log\lambda \neq o\sqrt{(lp/l_2p)}$.

We can choose μ so that $\mu \asymp \log\lambda\sqrt{(l_2p/lp)}$.

Then $\log\mu = O(l_2p)$, $a_2 \log 2 \log\mu = O(lp\, l_2p)$

$$\log\lambda/\pi(\mu) = O(\log\lambda/\mu.\ \log\mu) = O(lp\, l_2p)$$

$$\mu a_\lambda = O\left\{l\lambda.\ \sqrt{(l_2p/lp)}.\ (lp/l\lambda)\right\} = O\sqrt{(lp\, l_2p)}.$$

Hence from §6,

$$\frac{(t-1)\log\lambda}{\log\{1+(t-1)/a_\lambda\}} < \frac{(t-1)\log p}{\log t} + O\sqrt{(lp\, l_2p)} \quad (18)$$

and
$$\frac{(t-1)\log\lambda}{\log\{(t+a_\lambda)/(1+a_\lambda)\}} > \frac{(t-1)\log p}{\log t} + O\sqrt{(lpl_2p)}. \quad (19)$$

From (18),

$$\frac{\log t\log\lambda}{\log p} < \log\left(1+\frac{t-1}{a_\lambda}\right) + O\left(\sqrt{\frac{l_2p}{lp}}.\frac{1}{a_\lambda}\right)$$

i. e.
$$t^{\log\lambda/\log p} < 1 + \frac{t-1}{a_\lambda} + O\left(\sqrt{\frac{l_2p}{lp}}.\frac{1}{a_\lambda}\right).$$

* We shall write for brevity $lp = \log p$, $l_2p = \log\log p$, etc.

So, $$a_\lambda \leqslant \frac{t-1}{t^{\log\lambda/\log p}-1}+O\left\{\sqrt{\frac{l_2p}{lp}}\cdot\frac{1}{t^{\log\lambda/\log p}-1}\right\}$$

$$=(t-1)\;(t^{\log\lambda/\log p}-1)^{-1}+O\;\left(\frac{\sqrt{lpl_2p}}{l\lambda}\right) \qquad (20)$$

Similarly, (19) gives

$$a_\lambda+1\geqslant(t-1)\;(t^{\log\lambda/\log p}-1)^{-1}+O\left(\frac{\sqrt{lp\,l_2p}}{l\lambda}\right). \qquad (21)$$

8. Now we shall apply the above results to consider the behaviour of a_λ.

(i) $\log\lambda\neq O\sqrt{(lpl_2p)}$.

Let $B=(t-1)\;(t^{\log\lambda/\log p}-1)^{-1}$, and let e_λ be such that $B+e_\lambda$ is an integer; $-\frac{1}{2}<e_\lambda<\frac{1}{2}$, and $A=[B]$.

Arguing as in Ramanujan's memoir*, we obtain

$$a_\lambda=A,\text{ when } e_\lambda\neq o\;(\sqrt{(lpl_2p)}/l\lambda),$$

and a_λ is determined with an error not exceeding one, when $e_\lambda=o\;(\sqrt{(lpl_2p)}/l\lambda)$.

(ii) $\log\lambda=O\sqrt{(lpl_2p)}$, and $\log\lambda\neq o\sqrt{(lp/l_2p)}$.

Then all the conditions in §7 are satisfied.

But $B/(t-1)=lp/ltl\lambda+O(1)$. Hence

$$a_\lambda\log\lambda=(t-1)\;lp/lt+O\sqrt{(lpl_2p)}.$$

(iii) $\log\lambda=O\sqrt{(lp/l_2p)}$, and $\log\lambda\neq o(l_2p)$.

Then from (17), $a_\lambda\log\lambda=(t-1)\;lp/lt+O\sqrt{(lpl\lambda)}$.

(iv) $\log\lambda=O(l_2p)$. (17) gives

$$a_\lambda\log\lambda=(t-1)\;lp/lt+O\sqrt{(lpl_2p)}.$$

Thus we obtain fairly accurate information about a_λ as in the case of the highly composite numbers of Ramanujan.

9. Let λ and q be consecutive primes such that $\lambda<q<p$ and $\lambda^n<q^{n-2t+3}<\lambda^{n+1}$ where n is an integer. Assume further that $a_q>n-2t+3$. Then, taking $N'=N\lambda^n q^{-n+2t-3}$ we obtain

$$(a_\lambda+t-1)\;(a_q+t-1)>(a_\lambda+n+1)\;(a_q-n+2t-2).$$

* Paragraph 18, Range I.

Hence

$$a_\lambda(n-t+1) > a_q(n-t+1)+a_q-(n+1)(n-2t+2)-(t-1)^2.$$

Hence when $n \geqslant t$, $a_\lambda > a_q$ if $a_q > (n+1)(n-2t+2)+(t-1)^2$, i.e. in particular if $a_q > n^2$, since

$$n^2 \geqslant (n+1)(n-2t+2)+(t-1)^2 \text{ when } t \geqslant 2. \qquad (22)$$

Since $\lambda^{n+1} > q^{n-2t+3}$, we have $(n+1)\log\lambda > (n-2t+3) \times \log q$, i.e. $n\log(q/\lambda) < (2t-3)\log q+\log\lambda < 2(t-1)\log q$. Also, $\{\log(q/\lambda)\}^{-1} \leqslant \{\log(1+1/(q-1))\}^{-1} < 2(q-2) < 2q$, for $\log(1+x) \geqslant x/2$ when $0 \leqslant x \leqslant 1$.

From these, we get $n < 4(t-1)\, q\log q.$ (23)

Also from (5), $a_q > \dfrac{\log p}{\log q} - t + 1$. From (22) and (23), it follows that $a_\lambda > a_q$ if

$$\log p > 16(t-1)^2\, q^2(\log q)^3+(t-1)\log q.$$

So, if $16(t-1)^2\lambda^2(\log\lambda)^3+(t-1)\log\lambda < \log p$, we shall have $a_2 > a_3 > \ldots > a_\lambda$. In other words,

In a large highly composite number, the indices comparatively near the beginning form a strictly decreasing sequence.

10* We shall now write N in the form

$$2.3.5\ldots p_1.2.3.5\ldots p_2.2.3.5\ldots p_3\ldots,$$

the number of p's being a_2 and $p_1 > p_2 \geqslant p_3 \geqslant \ldots$ Let P_r be the prime next above p_r, so that

$$\log P_r = \log p_r + O(1). \quad \text{Then,}$$

$$\left.\begin{array}{l} a_{P_r} \leqslant r-1,\ a_p \geqslant r,\ a_P \leqslant a_{p_r}-1,\ a_{P_1}=0. \\ \text{From (5), it follows that} \\ a_{p_r}\log p_r \asymp \log p_1,\ (1+a_{P_r})\log P_r \asymp \log p_1. \\ \text{Hence also,}\quad r\log p_r \asymp \log p_1,\ a_{p_r} \asymp r. \end{array}\right\} \quad (24)$$

From these we get,

$$\log\left(1+\frac{-}{1+a_{P_r}}\right) \geqslant \log\left(1+\frac{t-1}{r}\right) \geqslant \log\left(1+\frac{t-1}{a_{p_r}}\right). \quad (25)$$

Putting $\lambda = p_r$ in (15) and $\lambda = P_r$ in (16), we have by (24), (16′) and the above results,

* Compare with para 9 of Ramanujan's memoir

$$\frac{(t-1)\log p_r}{\log\left(1+\frac{t-1}{r}\right)}=\frac{(t-1)\log p_1}{\log t}+O\left\{\frac{lp_1 l\mu}{\mu r}+\mu r+\surd(lp_1 l_2 p_1)\right\}. \tag{26}$$

(i) Let $r=o(\log p_1)$. Choose $\mu=o(lp_1/r)$, $\mu\neq O(1)$. Then (25) becomes

$$\frac{\log p_r}{\log\left(1+\frac{t-1}{r}\right)}\sim\frac{\log p_1}{\log t}.$$

Hence, *if* $r=o(\log p_1)$, *then*

$$\frac{\log p_1}{\log t}\sim\frac{\log p_2}{\log\left(1+\frac{t-1}{2}\right)}\sim\frac{\log p_3}{\log\left(1+\frac{t-1}{3}\right)}\sim\cdots$$

$$\sim\frac{\log p_r}{\log\left(1+\frac{t-1}{r}\right)}. \tag{27}$$

(ii) Let $r=O\surd(lp_1 l_2 p_1)$. Choose μ so that $\mu r\asymp\surd(lp_1\, l^2 p_1)$.

Then from (25)

$$\frac{\log p_r}{\log\left(1+\frac{t-1}{r}\right)}=\frac{\log p_1}{\log t}+O\surd(lp_1 l_2 p_1).$$

(iii) Similarly if $r\neq O\surd(lp_1 l_2 p_1)$, by taking $\mu=O(1)$, we have

$$\frac{\log p_r}{\log\left(1+\frac{t-1}{r}\right)}=\frac{\log p_1}{\log t}+O(r).$$

11. From (15), (16) and (25), we have

$$\log\left(1+\frac{t-1}{a_{p_r}}\right)\geqslant\frac{(t-1)\log p_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{\mu r}+\frac{\mu r}{\log p_1}\right)\right\}$$

and $$\log\left(1+\frac{t-1}{1+a_{P_r}}\right)\leqslant\frac{(t-1)\log P_r}{a_2\log 2}$$

$$\times\left\{1+O\left(\frac{\log\mu}{\mu r}+\frac{\mu r}{\log p_1}\right)\right\}.$$

Assuming $\mu r = o(\log p_1)$, we obtain by division

$$\frac{\log\left(1+\frac{t-1}{1+a_{P_r}}\right)}{\log\left(1+\frac{t-1}{a_{p_r}}\right)} \leqslant 1+O\left\{\frac{1}{\log p_r}+\frac{\log\mu}{\mu r}+\frac{\mu r}{\log p_1}\right\},$$

using $\log P_r = \log p_r + O(1)$. Hence $\log\left(1+\frac{t-1}{1+a_{P_r}}\right)$

$$\leqslant \log\left(1+\frac{t-1}{a_{p_r}}\right)+O\left\{\frac{1}{a_{p_r}\log p_r}+\frac{\log\mu}{\mu r a_{p_r}}+\frac{\mu r}{a_{p_r}\log p_1}\right\}$$

$$= \log\left(1+\frac{t-1}{a_{p_r}}\right)+O\left(\frac{\mu}{\log p_1}\right)+O\left(\frac{\log\mu}{\mu r^2}\right) \text{ from (24).}$$

Hence,

$$1+\frac{t-1}{1+a_{P_r}} \leqslant 1+\frac{t-1}{a_{p_r}}+O\left(\frac{\mu}{\log p_1}\right)+O\left(\frac{\log\mu}{\mu r^2}\right)$$

or
$$a_{p_r} \leqslant 1+a_{P_r}+O\left(\frac{\mu r^2}{\log p_1}\right)+O\left(\frac{\log\mu}{\mu}\right). \tag{28}$$

Let $r = o\sqrt{(lp_1)}$. We can choose μ so that $\mu r^2 = o(\log p_1)$, $\mu \neq O(1)$. Then the condition $\mu r = o(lp_1)$ is satisfied, and (28) gives $a_{p_r} \leqslant 1+a_{P_r}+o(1)$. But since a_{p_r} and a_{P_r} are integers, and in virtue of (24), we get, if $r = o\sqrt{(\log p_1)}$, then

$$a_{p_r} = 1+a_{P_r} = r.$$

Hence, *if* $r = o\sqrt{(\log p_1)}$, *then* $p_1 > p_2 > \dots > p_r$. *In other words, in a large highly composite number of the t th order, groups of equal indices occur comparatively near the end.*

12. Now, $d_t(\nu^{a_\nu}) = \frac{(a_\nu+1)\dots(a_\nu+t-1)}{(t-1)!}$. Hence

$$d_t(\nu^{a_\nu+\delta_\nu}) = d_t(\nu a^\nu).\left(1+\frac{\delta_\nu}{a_\nu+1}\right)\dots\left(1+\frac{\delta_\nu}{a_\nu+t-1}\right)$$

Arguing as in para 28 of Ramanujan's memoir, we see that two consecutive highly composite numbers of the tth order are asymptotically equal. Further, it follows that the number of highly composite numbers of the tth order not exceeding x is not of the form

$$o\left\{lx\sqrt{l_2 x}/(l_3 x)^{\frac{3}{2}}\right\}.$$

13. We shall next consider the nature of $d_t(N)$. From the form of N in § 10,

$$d_t(N) = t^{\pi(p_1)-\pi(p_2)}\left\{\frac{3\cdot 4\cdots(t+1)}{(t-1)!}\right\}^{\pi(p_2)-\pi(p_3)}\cdots$$

So, if q is the largest prime not exceeding a_2+t-1, and $d_t(N) = 2^{d_2}3^{d_3}\ldots q^{d_q}$, then

$$d_\lambda = \pi(p_{\lambda-t+1}) + O(p_{\lambda-t+2}) \text{ when } \lambda \geqslant t-1. \qquad (29)$$

Hence as $N \to \infty$, the number of prime factors as well as the indices in $d_t(N)$ tends to infinity. In particular, the equation $d_t(N) = k2^m$ has got only a finite number of solutions.

From (27) and (29), it follows that if $t \leqslant \mu < \lambda = o(\log p_1)$, then

$$\frac{d_\mu}{\log\left(1-\frac{t-1}{\mu}\right)} \sim \frac{d_\lambda}{\log\left(1-\frac{t-1}{\lambda}\right)}.$$

SUPERIOR HIGHLY COMPOSITE NUMBERS OF THE t TH ORDER.

14. A number N may be called a superior highly composite number of the t th order, if there is a positive number δ such that $d_t(N)/N^\delta \geqslant d_t(N')/N'^\delta$ for all $N'<N$, while $d_t(N)/N^\delta > d_t(N')/N'^\delta$ for all $N'>N$. Obviously all such numbers are highly composite of the t th order. If such a number N be written in the form $2^{a_2}3^{a_3}\ldots p_1^{a_{p_1}}$, we have from §2 that for all $N > N_0(t)$, $a_{p_1} = 1$. We shall assume hereafter that $N > N_0(t)$.

Let $N' = N/\lambda$, where $\lambda \leqslant p_1$. Then

$$\frac{(a_\lambda+1)\ldots(a_\lambda+t-1)}{\lambda^{\delta a_\lambda}} \geqslant \frac{a_\lambda\ldots(a_\lambda+t-2)}{\lambda^{\delta(a_\lambda-1)}}, \text{ i. e. } \lambda^\delta \leqslant \left(1+\frac{t-1}{a_\lambda}\right) \qquad (30)$$

Similarly, taking $N' = N\lambda$, we obtain $\lambda^\delta > \left(1+\frac{t-1}{1+a_\lambda}\right)$ (31)

Putting $\lambda = p_1$ and P_1 respectively in (30) and (31),

$$\frac{\log t}{\log P_1} < \delta \leqslant \frac{\log t}{\log p_1}$$

or $$p_1 \leqslant t^x < P_1, \text{ if } \delta = 1/x.$$

That is, p_1 is the largest prime not exceeding t^x.

From (30) and (31), we have

$$\frac{t-1}{\lambda^{1/x}-1} \geqslant a_\lambda > \frac{t-1}{\lambda^{1/x}-1} - 1.$$

Hence N is of the form

$$2^{\left[\frac{t-1}{2^{1/x}-1}\right]} 3^{\left[\frac{t-1}{3^{1/x}-1}\right]} \dots p_1,$$

where p_1 is the largest prime not exceeding t^x.

15. Putting λ equal to p_r and P_r respectively in (30) and (31), we have by (24),

$$\frac{t-1}{p_r^{1/x}-1} \geqslant a_{p_r} \geqslant r \geqslant 1 + a_{P_r} > \frac{t-1}{P_r^{1/x}-1}.$$

Hence, $$p_r \leqslant \{ 1+(t-1)/r \}^x < P_r.$$

p_r is therefore the largest prime not exceeding $\{ 1+(t-1)/r \}^x$. Hence N is of the form

$$2.3\dots p_1.\ 2.3\dots p_2.\ 2.3\dots p_3\dots,$$

where $p_1, p_2, \dots$ are the largest primes not exceeding t^x, $\{ (t+1)/2 \}^x, \dots$ respectively. Hence N is of the form

$$e^{\theta(t^x) + \theta\{(t+1)/2\}^x + \theta\{(t+2)/3\}^x + \dots}, \tag{32}$$

and $d_t(N)$ is of the form

$$t^{\pi(t^x)}.\ \{ (t+1)/2 \}^{\pi\{(t+1)/2\}^x}.\ \{ (t+2)/3 \}^{\pi\{(t+2/3)\}^x} \dots \tag{33}$$

for, $d_t(p^\alpha)/d_t(p^{\alpha-1}) = (t+\alpha-1)/\alpha$. (32) shows that for every value of $x \geqslant 1$, there corresponds one and only one value of N. From (32) and (33), we get

$$d_t(N) \leqslant N^{1/x}.\ \frac{t^{\pi(t^x)}}{e^{x^{-1}\theta(t^x)}}.\ \frac{\{ (t+1)/2 \}^{\pi\{(t+1)/2\}^x}}{e^{x^{-1}\theta\{(t+1)/2\}^x}} \dots \tag{34}$$

for all N and x; $d_t(N)$ is equal to the expression on the right side when N is given by (32).

16. From (34) and Ramanujan's result, viz.

$$\pi(x) \log x - \theta(x) = \int_2^x \frac{\pi(t)}{t}\, dt,$$

it follows that $\log d_t(N) \leqslant F(x)$, where

$$F(x) = \frac{1}{x}\log N.\frac{1}{x}\left\{\int_2^{t^x}\frac{\pi(y)}{y}dy + \int_2^{((t+1)/2)^x}\frac{\pi(y)}{y}dy + \ldots\right\}$$

for all N and x. Arguing as in paragraph 38 of Ramanujan's memoir, we get that $F(x)$ is a minimum when x is a function of N defined by the inequalities

$$\theta(t^y)+\theta\{(t+1)/2\}^y+\theta\{(t+2)/3\}^y+\ldots\begin{cases}<\log N, y<x\\ >\log N, y>x.\end{cases} \quad (35)$$

When x is the function of N so defined, let

$$D(N) = t^{\pi(t^x)}.\{(t+1)/2\}^{\pi\{((t+1)/2)\}^x}\ldots \quad (36)$$

Then from (34), we see that $d(N) \leqslant D(N)$ for all values of N, while $d(N) = D(N)$ for all superior highly composite numbers of the t th order. Thus $D(N)$ gives the maximum order of $d_t(N)$.

17. Now we shall apply known results about primes to find the maximum order of $d_t(n)$.

From (35)

$$\log N = \theta(t^x)+O\{(t+1)/2\}^x = \theta(t^x)+O(t^{xl}),$$

where $l = \log\{(t+1)/2\}/\log t$. Hence

$$\theta(t^x) = \log N + O(\log N)^l.$$

Similarly from (36), $\pi(t^x) = \log D(N)/\log t + O(\log N)^l$.

But* $\pi(x) = Li(x)+O(xe^{-a\sqrt{\log x}})$

$\theta(x) = x+O(xe^{-a\sqrt{\log x}})$.

Hence, $t^x = \log N + O(lNe^{-a\sqrt{l^2N}})$.

It follows that the maximum order of $d_t(N)$ is

$$t^{Li(\log N)+O(lNe^{-a\sqrt{l^2N}})}$$

18. We shall next determine the maximum order of $d_t(N)$ on the assumption of the truth of the Riemann hypothesis, according to which†

$$\pi(x) = Li\,x + O(x^{\frac{1}{2}}lx);\ \theta(x) = x+O(x^{\frac{1}{2}}(lx)^2).$$

From (35), $\log N = \theta(t^x)+\theta\left\{\left(\frac{t+1}{2}\right)^x\right\}+O\left(\frac{t+2}{3}\right)^x$

* Collected papers: p. 83. Landau, *Primzahlen*, p. 333.

† Landau: *Primzahlen*, p. 388.

$$= \theta(t^x) + \left(\frac{t+1}{2}\right)^x + O(t^{xh}), \tag{37}$$

where $h \log t = \log\{(t+2)/3\}$. From (36),

$$\log D(N) = \pi(t^x) \log t + Li\left(\frac{t+1}{2}\right)^x \log\frac{t+1}{2} + O(t^{xh}).$$

Writing X for t^x,

$$\log N = \theta(X) + X^k + O(X^h),$$

$$\log D(N) = \pi(X) \log t + Li\{X^k\} \log\frac{t+1}{2} + O(X^h),$$

where $k \log t = \log\{(t+1)/2.\}$ But from (37), $X = \log N + O(\log N)^k + O(\log N)^h$. Hence,

$$\log N = \theta(X) + (\log N)^k + O(\log N)^h.$$

$$\log D(N) = Li(X) \log t + \log\left(\frac{t+1}{2}\right) Li\{(\log N)^k\}$$
$$+ O(\log N)^h, \text{ when } t \geqslant 4$$
$$= \log t\, Li(\log N) + \log\left(\frac{t+1}{2}\right) Li(\log N)^k$$
$$- \log t (\log N)^k / l_2 N + O(\log N)^h.$$

Hence when $t \geqslant 4$, the maximum order of $d_t(N)$ is $t^{Li(\log N) + \phi_t(N)}$

where $\phi_t(N) = kLi(\log N)^k - (\log N)^k / l_2 N + O(\log N)^h$.

Of course it is theoretically possible to reduce the error term in $d_t(N)$ to the order of $\sqrt{(\log N)}$. But when t is large, other terms become complicated.

It can easily be shown that the maximum order of $d_3(N)$ is $3^{Li(\log N) + \phi_3(N)}$, where $\phi_3(N)$
$= gLi(\log N)^g - (\log N)^g / l_2 N - R(\log N) + O\{\sqrt{(lN)}/(l_2 N)^3\}$,
where $g = \log 2/\log 3$, and $R(x)$ is given by (220) of Ramanujan's memoir.

ON THE DIFFERENTIATED SERIES OF EIGENFUNCTIONS.

BY

S. MINAKSHISUNDARAM.

[Received 17 April, 1944.]

1. Let $f(x,y)$ be a function integrable L^2 in a bounded closed domain D whose boundary C is composed of a finite number of regular curves and let

$$f(x,y) \sim \Sigma a_n \omega_n(x,y), \tag{1}$$

where $\omega_n(x,y)$ are the eigenfunctions, with the corresponding eigenvalues μ_n, of the Green's function for D vanishing on C. If (x,y) is a point interior to D and C_r is a circular region with centre (x,y) and radius r, lying entirely in D, we set

$$\phi(r) = \int_0^{2\pi} f(x+r\cos\theta,\ y+r\sin\theta)\cos\theta\, d\theta \tag{2}$$

$$\psi(r) = \int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\sin\theta\, d\theta \tag{3}$$

$$\theta_p(x,y;r) = \frac{p}{\pi r^2}\iint_{C_r}\left(1-\frac{t^2}{r^2}\right)^{p-1} f(\xi,\eta)\, d\xi\, d\eta, \tag{4}$$

where $(\xi-x)^2+(\eta-y)^2 = t^2 \leqslant r^2$. The main theorem of this paper is

THEOREM A. *If* $\phi(r)\to\alpha$ *as* $r\to 0$, $\lim\limits_{t\to 0} t^{\frac{1}{2}}\Sigma a_n \partial\omega_n/\partial x\, e^{-\mu_n t} = \frac{1}{4}\pi^{-\frac{1}{2}}\alpha$ *and if* $\psi(r)\to\beta$ *as* $r\to 0$, $\lim\limits_{t\to 0} t^{\frac{1}{2}}\Sigma a_n\partial\omega_n/\partial y\, e^{-\mu_n t} = \frac{1}{4}\pi^{-\frac{1}{2}}\beta$. *But if* $\phi(r)/r\to\gamma$ *as* $r\to 0$, *then* $\lim\limits_{t\to 0}\Sigma a_n\partial\omega_n/\partial x\, e^{-\mu_n t} = \gamma/\pi$ *and if* $\psi(r)/r\to\delta$ *as* $r\to 0$, *then* $\lim\limits_{t\to 0}\Sigma a_n\partial\omega_n/\partial y\, e^{-\mu_n t} = \delta/\pi$.

2. It is easily observed that

$$\theta_p(r) \equiv \theta_p(x,y;r) = \frac{2p(p-1)}{r^4}\int_0^r\left(1-\frac{t^2}{r^2}\right)^{p-2} t^3\theta_1(t)\, dt \tag{5}$$

and

$$\frac{\partial\theta_1}{\partial x}=\frac{\phi(r)}{\pi r}, \tag{6}$$

$$\frac{\partial\theta_1}{\partial y}=\frac{\psi(r)}{\pi r} \tag{7}$$

whenever both sides of (6) and (7) exist. So if p is large enough, we can always write

$$\frac{\partial\theta_p}{\partial x}=\frac{2p(p-1)}{\pi r^4}\int_0^r(1-t^2/r^2)^{p-2}t^2\phi(t)\,dt \tag{8}$$

with a similar expression for $\partial\theta_p/\partial y$.

Lemma 1. *If* $\phi(r)\to\alpha$, *then* $\dfrac{\partial\theta_p}{\partial x}\sim\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2})}\dfrac{\alpha}{2\pi^{\frac{1}{2}}r}$, *as* $r\to 0$.

Lemma 2. *If* $\dfrac{\phi(r)}{r}\to\gamma$, *then* $\dfrac{\partial\theta_p}{\partial x}\to\dfrac{\gamma}{\pi}$, *as* $r\to 0$.

These follow from (8) and we can prove corresponding lemmas for $\psi(r)$.

Lemma 3. $\dfrac{\partial\omega_n}{\partial x}=O\left(\lambda_n^{\frac{7}{4}}\right),\dfrac{\partial\omega_n}{\partial y}=O\left(\lambda_n^{\frac{7}{4}}\right)$ *where* $\lambda_n=\mu_n^{\frac{1}{2}}$ *and* (x, y) *is a point interior to* D.

This is due to Hammerstein*.

Now if $f(x,y)\ \varepsilon\ L^2$ in D we observe that $\Sigma a_n\,\alpha_p(\lambda_n r)\partial\omega_n/\partial x$† is absolutely convergent for every $r>0$ if p is large enough. In fact, using Schwarz's inequality and Lemma 3, we have

$$\sum\left|a_n\frac{\partial\omega_n}{\partial x}\alpha_p(\lambda_n r)\right|\leqslant\left[\sum a_n^2\sum\left\{\frac{\partial\omega_n}{\partial x}\alpha_p(\lambda_n r)\right\}^2\right]^{\frac{1}{2}}$$
$$\leqslant O\left(\Sigma 1/\lambda_n^{2p-5/2}\right)<\infty\,,\ p>\tfrac{9}{4}.$$

Hence we have

Lemma 4. *If* $f(x,y)\ \varepsilon\ L^2$ *in* D *and* $f(x,y)\sim\Sigma a_n\omega_n$, *then for* $p>\frac{9}{4}$, $\Sigma a_n\,\alpha_p(\lambda_n r)\,\partial\omega_n/\partial x$ *is absolutely convergent for every* $r>0$. *The sum is bounded and of the order* $r^{-p-\frac{1}{2}}$ *as* $r\to\infty$, *while for sufficiently small values of* r, *it is equal to* $\partial\theta_p/\partial x$.

Now, let us write for $p>9/4$

$$\vartheta_p(x,y\,;\,r)\equiv\vartheta_p(r)\equiv\Sigma a_n\,\alpha_p(\lambda_n r)\,\partial\omega_n/\partial x. \tag{9}$$

* *Math. Annalen,* 93 (1925), 102-9.

† $\alpha(x)=2^p\Gamma(p+1)\,J_p(x)/x^p$.

Multiplying both sides of (9) by

$$2r^{2p+1}\exp\,(-r^2/4t)/(4t)^{p+1}\Gamma(p+1)$$

and integrating with respect to r from 0 to ∞, we observe that

$$\Sigma a_n \frac{\partial \omega_n}{\partial x}\exp(-\mu_n t) = \frac{2}{(4t)^{p+1}\Gamma(p+1)}\int_0^\infty r^{2p+1}\exp\left(-\frac{r_2}{4t}\right)\vartheta_p(r)\,dr \tag{10}$$

provided the right side exists. If we now apply Lemma 1, we observe that when $\phi(r)\to\alpha$, $\vartheta_p(r)\equiv\partial\theta_p/\partial x$ for small r, is of the order $1/r$ as $r\to 0$, the integral on the right of (10) exists for every $t>0$, on acccount of Lemma 4, and as $t\to 0$ it is asymptotically equivalent to

$$\frac{2}{(4t)^{p+1}\Gamma(p+1)}\frac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2})\ 2\sqrt{\pi}}\int_0^\infty r^{2p}\exp\left(-\frac{r^2}{4t}\right)dr = \frac{\alpha}{4(\pi t)^{\frac{1}{2}}}.$$

If however we apply Lemma 2 we note that the right side of (10) tends as $t\to 0$ to

$$\frac{2\gamma}{\pi(4t)^{p+1}\Gamma(p+1)}\int_0^\infty r^{2p+1}\exp\left(-\frac{r^2}{4t}\right)dr = \frac{\gamma}{\pi}.$$

It is clear that starting with $\psi(r)$ we may prove analogous results for $\Sigma a_n \dfrac{\partial\omega_n}{\partial y}\exp(-\mu_n t)$ in virtue of (7).

Proof of Theorem A is thus complete.

4. It is indeed possible to prove results corresponding to those in Theorem A, using summability by typical means. One has only to multiply both sides of (9) by $w^{2p+2}\,r^{2p+1}\alpha_q(wr)$, $q>2p+\frac{3}{2}$, $p>\frac{9}{4}$; integrate with respect to r from 0 to ∞ and then let $w\to\infty$. We now state the result:

THEOREM B:—*If $\phi(r)\to\alpha$ as $r\to 0$, then for large k, say* $k>13/4$

$$\lim_{w\to\infty}\frac{1}{w}\sum_{\lambda_n\leqslant w} a_n\frac{\partial\omega_n}{\partial x}\left(1-\frac{\lambda_n^2}{w^2}\right)^k = \frac{\Gamma(k+1)\alpha}{4\pi^{\frac{1}{2}}\Gamma(k+\frac{3}{2})}$$

and if $\phi(r)/r\to\gamma$ then

$$\lim_{w=\infty}\sum a_n\frac{\partial\omega_n}{\partial x}\left(1-\frac{\lambda_n^2}{w^2}\right)^k = \gamma/\pi.$$

Similarly if $\psi(r) \to \beta$ *or* $\psi(r)/r \to \delta$ *we shall have corresponding results for* $\Sigma a_n \partial\omega_n/\partial y$.

5. As a consequence of Theorems A and B it may be noted that at a point where $f(x, y)$ is differentiable, $\Sigma a_n \partial\omega_n/\partial x$ and $\Sigma a_n \partial\omega_n/\partial y$ are summable by typical means to the derivatives of f, and the two Dirichlet's series $\Sigma a_n \partial\omega_n/\partial x \exp(-\mu_n t)$ and $\Sigma a_n \partial\omega_n/\partial y \exp(-\mu_n t)$ tend, as $t \to 0$, to the partial derivatives of $f(x, y)$.

NOTES ON GROUP-THEORY IV—VI.

BY

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[Received 20 June, 1944].

IV: THE GROUPS $[A, B]$.

This note deals with groups which are formed by two groups, say A and B, using certain homomorphisms. The resulting group $[A, B]$ will be shown to be commutative and to depend on $A/C(A)$ and $B/C(B)$ only. When the two factorgroups admit representation by finite bases, a similar representation of $[A, B]$ can be derived (§3). Some results can be generalized for arbitrary groups A and B. Applications of the groups $[A, B]$ will be made in some of the subsequent notes.

1. Let A and B be two groups the elements of which will be denoted by the corresponding small letters with and without indices. Construct another group $[A, B]$ by the generators $[a_i, b_k]$ and the generating relations

$$[a_l a_k, b] = [a_l, b][a_k, b], \quad [a, b_\mu b_\nu] = [a, b_\mu][a, b_\nu] \quad (1)$$

which are supposed to hold for all the a-elements and all the b-elements.

Hence if b is a fixed element and a_i runs over A, then the elements $[a_i, b]$ form a subgroup $[A, b]$ of $[A, B]$ which is homomorphic to A. Similarly for every a, there is a subgroup $[a, B]$ which is homomorphic to B. Let a_0, b_0 and 1 be the unit elements of the groups A, B and

$[A, B]$ respectively. Then

$$[a, b] = [a.a_0, b] = [a, b]\ [a_0, b]$$
$$= [a, b.b_0] = [a, b]\ [a, b_0].$$

Hence $$[a_0, b] = [a, b_0] = 1 \qquad (2)$$

for every a and every b. Moreover

$$[a_j a_k, b_\mu b_\nu] = [a_j a_k, b_\mu]\ [a_j a_k, b_\nu]$$
$$= [a_j, b_\mu]\ [a_k, b_\mu]\ [a_j, b_\nu]\ [a_k, b_\nu]$$
$$= [a_j, b_\mu b_\nu]\ [a_k, b_\mu b_\nu]$$
$$= [a_j, b_\mu]\ [a_j, b_\nu]\ [a_k, b_\mu]\ [a_k, b_\nu].$$

Hence $$[a_k, b_\mu]\ [a_j, b_\nu] = [a_j, b_\nu]\ [a_k, b_\mu]. \qquad (3)$$

The generators of $[A, B]$ are therefore commutative; thus the group is *Abelian*. Hence $[a.a_i a_k a_i^{-1} a_k^{-1}, b] = [a, b]$. Thus in $[a, b]$, the element a can be replaced by any other element of the coset $aC(A)$ of the commutator-group $C(A)$ of A. Similarly b can be replaced by every element of $bC(B)$. Hence we have the isomorphism

$$[A, B] \cong [A/C(A), B/C(B)]. \qquad (4)$$

From (1) it follows that

$$[a, b]^m = [a^m, b] = [a, b^m]. \qquad (5)$$

Let r be the order of a and s the order of b; moreover let there exist elements a_* and b_* such that $a_*^\sigma = a$ and $b_*^\rho = b$ where ρ is a factor of r and σ a factor of s. Then $[a, b] = [a_*^\sigma, b] = [a_*, b^\sigma] = [a^\rho, b_*]$. Hence $1 = [a, b]^{r/\rho} = [a, b]^{s/\sigma}$ and therefore

$$[a, b]^t = 1, \text{ where } t = (r/\rho, s/\sigma). \qquad (6)$$

The groups A and B may have common elements, say c and d; then $[c, d]$ and $[d, c]$ are in general different elements. The groups $[A, B]$ and $[B, A]$ are different unless $A = B$, but from (2), it follows that they are isomorphic. Moreover it follows in the same way that

$$[[A, B], C] \cong [A, [B, C]]. \qquad (7)$$

We may denote a group of the type by $[A, B, C]$ and by an easy generalization of the conditions (2) obtain groups $[A_1, A_2, \ldots, A_n]$.

Let $A = A_1 \times A_2$, then for every element a_{ik} of A there is $a_{ik} = a_i^1 a_k^2 = a_k^2 a_i^1$ where a_i^1 and a_k^2 are uniquely determined elements of A^1 and A^2 respectively. The group $\{A, B\} = [A_1, B] \times [A_2, B]$ is generated by the elements

$$[a_i^1, b]\,[a^2_k, b] = \{a_{ik}, b\}$$

and for these elements conditions corresponding to those of (2) hold, the brackets [] being replaced by $\{\ \}$. Hence $[a_{ik}, b] \to \{a_{ik}, b\}$ is a homomorphism. However $\{a_{ik}, b\} = 1$ implies $[a^1_i, b] = [a^2_k, b] = 1$ and therefore $[a_{ik}, b] = 1$. Hence the homomorphism is an isomorphism. Hence

$$[A^1 \times A^2, B] \simeq [A^1, B] \times [A^2, B].$$

The corresponding result holds for B. By mathematical induction we therefore obtain the general formula

$$[\Pi A^i, \Pi B^k] \simeq \Pi_{i,k}[A^i, B^k], \tag{8}$$

where the products are taken as direct products. The groups therefore form something like commutative rings* (in the sense of isomorphism). Here direct multiplication stands for "addition" and the operation [ ] for multiplication", but neither of the two operations is unconditionally reversible. The result of the operation [ ] is always an Abelian group and depends only on the factor-groups of the commutatorgroups of the group concerned.

2. To investigate the groups $[A, B]$ there is no loss of generality in supposing that A and B are both Abelian groups. Suppose moreover that each of the groups has a finite basis, then they are direct products of cyclic groups A^i, B^k which are either of orders p^m, where p is a prime

* The author regrets that he has no access to two recent papers apparently dealing with the theory of ideals of such systems : Fritz Klein-Barmen *Über gewisse Halbverbände und kommutative Semigruppen I und II*, *Math. Z.* 48 (1942) 275-88 and (1943) 715-34. See *Mathematical Reviews*, 5 p. 31.

number, or of infinite order. Hence it follows from (8) that

$$[A, B] = \prod_{i, k} [A^i, B^k].$$

1. Let A^i and B^k be cyclic groups of order p^m and q^n respectively where p and q are different prime numbers. Then to every element b of B^k there exists an element b_* for which $b_*^{p^m} = b$; hence from (6) we get $[A^i, B^k] = 1$.

2. Let A^i and B^k be cyclic groups of order p^m and p^n respectively where $m \leqq n$. Then the elements of A^i can be represented in the form a^μ, where $0 \leqq \mu < p^m$, the elements of B^k in the form b^ν where $0 \leqq \nu < p^n$ and therefore the elements of $[A^i, B^k]$ by $[a, b]^t$. As $[a, b]^{p^m} = 1$, the group is homomorphic to a cyclic group of order p^m. On the other hand if we map $[a^\mu, b^\nu] \rightarrow c^{\mu\nu}$, where c is an element of order p^m, the conditions (2) are satisfied. Hence $[A^i, B^k]$ is a cyclic group of order p^m. In the same way we prove:

3. If A^i is a cyclic group of order p^m and B^k an infinite cyclic group, then $[A^i, B^k]$ is a cyclic group of order p^m.

4. If A^i and B^k are infinite cyclic groups, then $[A^i, B^k]$ is an infinite cyclic group.

If in particular A, B and therefore $[A, B]$ are finite Abelian groups, then $[A, B]$ is the direct product of groups $P = [P', P'']$, where P, P', P'' consist of those elements of the three groups respectively which belong to the same prime number p (Abelian p-groups). Represent P, P', P'' as direct products of cyclic groups; let there be exactly t_m, t'_m, t''_m direct factors respectively of order p^m, for $m = 1, 2, \ldots$ and s_m, s'_m, s''_m direct factors respectively of order $\geqq p^m$. The order of a direct factor $P_{ik} = [P'_i, P''_k]$ of P is $\geqq p^m$ if and only if the order of both the groups

P'_i and P''_k, which are direct factors of P' and P'' respectively, is $\geqq p^m$. Hence

$$s_m = s'_m s''_m \text{ for } m = 1, 2 \ldots \tag{9}$$

$t_m = s_m - s_{m+1}$, $t'_m = s'_m - s'_{m+1}$, $t''_m = s''_m - s''_{m+1}$ and therefore
$s_1 \geqq s_2 \geqq \ldots \geqq 0$; $s_1' \geqq s_2' \geqq \ldots \geqq 0$; $s_1'' \geqq s_2'' \geqq \ldots \geqq 0$. (10)
On the other hand if (9) and (10) are satisfied, the characteristic numbers t_m, t'_m, t_m'' are uniquely determined and for the corresponding groups P, P', P'' holds: $P \sim [P', P'']$.

The finite abelian p-groups are therefore completely characterised by diagonal matrices

$$\begin{Vmatrix} s_1 & & & & & \\ & s_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & s_r \end{Vmatrix}$$

with a monotonic decreasing diagonal of integral non-negative elements and the composition of the groups by the operation [ ] is performed by the multiplication of the corresponding matrices. In this way we can find out whether a given finite Abelian p-group can be decomposed by the operation [ ] into simpler p-groups. If, e.g. s_1 is a prime number, the group is *irreducible*, but this condition is not necessary as shown by the example $s_1 = 4$, $s_2 = 3$. The decomposition into irreducible groups is always possible but in general not unique. E.g.

$$\begin{pmatrix} 15 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

3. Let A and B be Abelian groups with finite bases, say

$$a_{01}, \ldots a_{0m}, a_{jk1}, \ldots, a_{rtn}.$$
$$b_{01}, \ldots b_{0\mu}, b_{k\lambda 1}, \ldots, b_{\rho\tau\nu}.$$

Here the index zero indicates that the basis element is of order zero,* whereas a basis element with the first two indices j, k (κ, λ) is of order p_j^k, where p_i is the j th prime number. Now

$$a = \prod_{i=1}^{m} a_{0i}^{\;s_i} \prod_{i=1}^{n} a_{jki}^{\;u_i}, \quad b = \prod_{i=1}^{\mu} b_{0i}^{\;\sigma_i} \prod_{i=1}^{\nu} b_{k\lambda i}^{\;v_i}, \qquad (11)$$

where s_i and σ_i may take any integral values whereas $0 \leqq u_i < p_j^k$, $0 \leqq v_i < p_\kappa^\lambda$. Then $[a, b] =$
$\prod_{i,t} [a_{0i}, b_{0t}]^{s_i\sigma_t} \prod_{i,t} [a_{0i}, b_{k\lambda t}]^{s_i v_t} \prod_{i,t} [a_{jki}, b_{0t}]^{u_i\sigma_j} \prod_{j} \prod_{i,t} [a_{jki}, b_{j\lambda t}]^{u_i v_t}$.

In the last set of produtcts only those third indices i and t must be used which occur in connection with the index j. The order of $[a, b]$ is the lowest common multiple of the orders of the direct factors.

Here *order* $[a_{0i}, b_{0t}]^{s_i\sigma_t} = 0$ or 1 according as $s_i\sigma_t \neq 0$, or $= 0$

order $[a_{0i}, b_{\kappa\lambda t}]^{s_i v_t} = p_\kappa : (p_\kappa, s_i v_t)$ (12)

order $[a_{jki}, b_{0t}]^{u_i\sigma_t} = p_j^k : (p_j^k, u_i\sigma_t)$

order $[a_{jki}, b_{j\lambda t}]^{u_i v_t} = p_j^w : (p_j^w, u_i v_t)$, where w is the smaller of the two integers k and λ.

If A and B are both cyclic groups, the order of $[a, b]$ is uniquely determined by the orders of a and b and by the exponents for which the equations $x^n = a, y^m = b$ can be solved. In general the order of $[a, b]$ does not depend on these properties alone as shown by the following example. A has the basis a_1, a_2 where *order*$(a_1) = 0$, *order* $(a_2) = p$. B is generated by b of order p^2. Then $[a_1^p a_2, b^p] = [a_1, b]^{p^2}[a_2, b]^p = 1$, whereas $[a_1 a_2, b^p] = [a_1, b]^p$ is of order p. However $a_1^p a_2$ and $a_1 a_2$ are both of order 0 and $x^m = a_1^p a_2$ as well as $x^m = a_1 a_2$ is solvable

* If a group element a generates an infinite cyclic group, it is often convenient to assign to it the order 0. (See H. Prüfer, *Math. Zeit.* 20,165-87). The integers m for which $x^m = a$ has a solution. form a (principal) ideal (m) in the ring of the integral number. I, one calls m the order of a, the definition tallies for $m \neq 0$ with the usual one, whereas $m = 0$ is the case of an infinite cycle.

for $m = \pm 1$ only. Thus the order of $[a, b]$ does not depend on a and b alone nor is the homomorphism $A \to [A, b]$ completely determined by A and the properties of b. Nevertheless it is possible to derive from (11) and (12) some statements which can be generalized to the case where A and B do not have a finite basis.

THEOREM 1. *$[a, b]$ is of order* 0 *if and only if a and b are both of order* 0.

PROOF. If a and b are both of order 0, then at least one exponent s_i and one σ_t are different from 0 and therefore one of the factors in (11) is of order 0; hence the *l.c.m.* is equal to zero. If however either a or b is of order > 0, then the factors in (11) have all orders > 0, and the *l.c.m.* is > 0.

THEOREM 2. *If the prime number p is a factor of order (a) and of order (b), but neither $x^p = a$, nor $y^p = b$ have solutions, then p is a factor of order $[a, b]$.*

PROOF. The order of at least one of the direct factors of $[a, b]$ in (11) is divisible by p.

THEOREM 3. *If A contains only elements of order $\leqq 1$, $x^p = a$ has no solution, and order (b) is divisible by p, then p is a factor of order $[a, b]$.*

PROOF. At least one exponent s_i in (11) is prime to p. Moreover we can suppose without loss of generality that *order* $(b) \neq 0$; therefore when $p = p_k$, for at least one basis element $b_{\kappa\lambda t}$, there is $0 < v_t < p^\lambda$. Hence *order* $[a_{0i}, b_{\kappa\lambda t}]^{s_i v_t}$ is divisible by p and so is *order* $[a, b]$.

4. Let A and B be arbitrary Abelian groups and let *order* $[a, b] = m$. The relation $[a, b]^m = 1$ depends on a finite number of relations (1). In these relations only a finite number of a-elements and of b-elements occur. These a-elements together with a generate a subgroup A' of A with a finite basis, and similarly b and the b-elements generate a subgroup B' with a finite basis. Hence *order* $[a, b] = m$ even if a is considered as an element of A' and

b as an element of B'. For this reason *Theorems* 1, 2 *and* 3 *hold also when A and B are arbitrary Abelian groups.* The following are consequences of the last statement.

THEOREM 4. *If every element of A is of order $\leqslant 1$ and b is of order* 0, *then $[A, b]$ is isomorphic to A.*

PROOF. If not so, there exists a relation $R[a_i, b] = 1$ whereas $R(a_i) = a \neq a_0$. Hence *order* $a = 0$ and therefore $R[a_i, b] = [a, b]$ is of order 0, in contradiction to the supposition.

THEOREM 5. *If every element of A is of order $\leqq 1$ and b is of order $m > 1$, then $[A, b] = 1$ if and only if the equation $x^m = a$ has a solution for every element a of A.*

PROOF. If $x^m = a$ has a solution, then $[a, b] = [x, b^m] = 1$. If however there exists an element a for which the equation has no solution in A, then there exists at least one prime number power p^n which is a factor of m and for which $x^{p^n} = a$ has no solution in A. Let $k = p^t$ be the highest power of p for which $x^k = a$ has a solution in A say $x = a'$, then $x^p = a'$ has no solution in A and, for Theorem 3, *order* $[a', b]$ is divisible by p and therefore $\neq 1$.

If A is the direct product of n infinite cyclic groups and B is a cyclic group of order m, then $[A, B]$ is the direct product of m cyclic groups of order m. We can extend A to an Abelian group A^* in which $x^m = a$ is solvable for every a of A^*, then $[A^*, B] = 1$. This shows a fundamental difference between groups which have a finite basis and those which have not. Suppose A and B have both finite bases, then it follows immediately from (11) and (12) that $[A, B] = 1$ if and only if either *order* $A = 1$ or *order* $B = 1$ or both the groups are finite and their orders are relatively prime. If therefore $[A, B] \neq 1$ and we extend A to $\bar{A}$, B to $\bar{B}$ such that both the new groups have finite bases, then $[\bar{A}, \bar{B}] \neq 1$. The following

example*—which can easily be generalized—seems to be interesting in this connection.

Let $\alpha_3, \alpha_4, \ldots$ be an infinite non-periodic sequence of integral numbers $0 < \alpha_i < p$, where p is a prime number; moreover $a_1, a_2, \ldots$ are the generators of an Abelian group A with the relations $a_i^p = a_{i-1}a_1^{\alpha_i}$ for $i = 3, 4 \ldots$. B is a cyclic group of order p generated by b. Then $1 = [a_i, b]^p = [a_{i-1}, b][a_1, b]^{\alpha i}$ and therefore for $j = 2, 3, \ldots$ and $k = 1, 2, \ldots, p$, $[a_j, b^k] = [a_1, b]^{-k\alpha_j+1}$. Hence the group $[A, B]$ consists of the elements $[a_1, b^k]$ only; it is therefore homomorphic to a group of order p and as it contains more than one element (by Th. 5) it is isomorphic to it. As the sequence $\alpha_3, \alpha_4, \ldots$ is supposed to be non-periodic, for each element a of A the equations $x^{p^m} = a$ are solvable for a finite number of exponents m only. This follows from a theorem of K. Hensel† that the p-adic representation of a rational number is necessarily periodic.

V. GENERALIZED PRODUCTS OF GROUPS.

Let A and B be any pair of groups having only the group unit 1 in common, $A*B$ their free products; then a *generalized* product $A \circ B$ of the groups A and B will be created by introducing additional relations into $A*B$ which satisfy the following two conditions.

1. Every additional relation can be derived from a set of *generalized relations* $f_\nu(x_1, \ldots, x_m; y_1, \ldots, y_n)$ in such a way that $f = 1$ holds when $x_1, \ldots, x_m$ are replaced by arbitrary elements of any one of the two groups A, B and $y_1, \ldots, y_n$ are replaced by arbitrary elements of the other group.

2. The homomorphism $A*B \to A \circ B$ is an isomorphism for A and for B.

If, e.g. there is only the generalized relation $xyx^{-1}y^{-1}$, the generalized product becomes the direct product $A \times B$,

* See the author's pamphlet "*Abelsche Gruppen mitabzählbaren Elementen*" Leipzig 1917.

† K. Hensel, *Theorieder algebraischen Zahlen*, 1 (1908) p. 39.

In general, when replacing $x_1, \ldots, x_m$ or $y_1, \ldots, y_n$ all by 1; then we obtain relations $f_\nu(1; y_1, \ldots, y_n) = 1$ and $f_\nu(x_1, \ldots, x_m; 1) = 1$ which must hold in A and in B. Adjoin to the relations of AoB the condition that all the elements of A are commutative with those of B, then $f_\nu = 1$ can be replaced by a relation where the x symbols are interchanged with the neighbouring y symbols, e.g. by $f_\nu(x_1, \ldots, x_m; 1) \cdot f_\nu(1; y_1, \ldots, y_n) = 1$. But this relation can be omitted since it is known to be satisfied. Hence by adjoining the condition of commutativity, AoB is transformed into $A \times B$. Thus:

THROREM. *$A \times B$ is homomorphic to AoB and AoB is homomorphic to $A * B$.*

So it appears that the direct product and the free product are extreme cases of general products. In any particular kind of general products, two generalized relations $f_1(x_1, \ldots, x_m; y_1, \ldots, y_n)$ and $f_2(x_1, \ldots, x_p; y_1, \ldots, y_q)$ can be replaced by a single one $f(x_1, \ldots, x_{m+p}; y_1, \ldots, y_{n+q}) = f_1(x_1, \ldots, x_m; y_1, \ldots, y_n) f_2(x_{m+1}, \ldots, x_{m+p}; y_{n+1}, \ldots, y_{n+q})$. Therefore any finite number of generalized relations can be replaced by a single one. At present, no example is known to the author of general products which need an infinity of generalized relations for their representation.

If A' is a subgroup of A and if B' is a subgroup of B, then the subgroup of AoB which is generated by A' and B' is homomorphic but not necessarily isomorphic to $A'oB'$. Define $G(A, B, C)$ when the 3 groups and the generalized relations are given, by supposing that $f_\nu = 1$ whenever the x-symbols are replaced by elements of one of the 3 groups and the y-symbols by any other one. Then $(AoB)oC$ and $Ao(BoC)$ are homomorphic but not necessarily isomorphic to $G(A, B, C)$. In special cases of general products (e.g. free products, direct products and the S-products) the general multiplication of groups satisfies the associative law, whereas the commutative law $AoB = BoA$ does hold in every case. The

operation $[A, B]$ of the preceding note is not a general product.

VI. *S*-PRODUCTS.

Consider the generalized products determined by the generalized condition

$$(x_1, (x_2, y)). \tag{1}$$

Thus in the general product $A \circ B$ the commutators of an a-and a b-element are supposed to be commutative with every a-and with every b-element and therefore lie in the centre of the group which will be called the *S-product* of A and B. Denote with a resp. b the elements of A resp. B. Then $(a^{-1}, b) = a^{-1}(b, a)a = (a, b)^{-1}$ and therefore $(a_1 a_2, b) = a_1(a_2, b)\ (a_1{}^{-1}b)^{-1}a_1{}^{-1} = (a_1, b)$ (a_2, b). Similarly $(b_1 b_2, a) = (b_1 a)(b_2, a)$. Thus it follows from Note IV that the group generated by the commutators (a, b) is homomorphic to the group $[A, B]$ where the elements $[a, b]$ of $[A, B]$ are mapped on the corresponding commutators (a, b). Hence every element can be represented in the form

$$a\, b\gamma, \tag{2}$$

where γ is an element of $[A, B]$ and

$$(a_1 b_1 \gamma_1)\ (a_2 b_2 \gamma_2) = a_1 a_2 b_1 b_2 \gamma,$$

where

$$\gamma = \gamma_1 \gamma_2\, [a_2 b_1{}^{-1}]. \tag{3}$$

To show that the homomorphism is an isomorphism, one must only show that (3) defines a group composition for the elements (2) and that in the group so determined, the generalized condition (1) is satisfied, A and B being subgroups of that group. It is easy to verify the associative law for (3) and the other statements are obvious. Hence:

THEOREM 1. *The elements of the S-product $A \circ B$ admit a unique repre entation in the form $ab\gamma$, where a runs over A, b over B and γ over $[A, B]$ independently and the rule of composition is given by* (3).

From Note IV, §1 it follows that *if* a belongs to the commutatorgroup $C(A)$ of A, then $[a,b]=1$; hence in the S-product $(a,b)=1$, i.e. the elements of $C(A)$ are commutative with every element of B. The subgroup of $A\circ B$ generated by $C(A)$ and B is isomorphic to $C(A)\times B$ and only homomorphic to $C(A)\circ B$. Consider three groups A, B, C. As the elements of $[A,B]$ belong to $C(A\circ B)$, they are commutative with all the elements of C in the group $D=(A\circ B)\circ C$. Hence the elements of D can be represented in the form $abc\alpha\beta\gamma$ where $a,b,c,\alpha,\beta,\gamma$ run independently over $A,B,C,[B,C],[A,C],[A,B]$. Moreover the elements of the three latter groups lie in the centre of D; and it is $(a,b)=[a,b]$, $(b,c)=[b,c]$, $(a,c)=[a,c]$. On the other hand we can define a group D' by these conditions and then verify by a simple calculation that D' satisfies the generalized condition $(x_1,(x_2,y))$ when x_1 and x_2 run over all the elements of $A\circ B$ and y runs over C, as well as when y runs over $A\circ B$ and x_1, x_2 run over C. Hence D is not only homomorphic but isomorphic to D' and by symmetry the same statement holds for $A\circ(B\circ C)$. Hence:

THEOREM 2. *The S-products satisfy the associative law* $A\circ(B\circ C)=(A\circ B)\circ C$.

If the elements of $A/C(A)$ and of $B/C(B)$ are all of order* $\leqslant 1$, then the same holds for the elements of $[A,B]$ as has been shown in Note IV Th.5.

THEOREM 3. *If A and B are ordered groups and the elements of $A/C(A)$ and $B/C(B)$ are all of order $\leqslant 1$, then the S-product $A\circ B$ can be ordered.*

PROOF. Since $[A,B]$ is an Abelian group with elements of order $\leqslant 1$ only, this subgroup can be ordered†. Let $[A,B]$ be ordered in any admissible way and let

* As in Note IV, the arithmetical notation of order is used, putting order O instead of order ∞.

† About ordered group, see the author's notes, *Proc. Ind. Ac. of Sc.* 16 (1942) 256-65, 17 (1943) 199-201.

$ab\gamma \neq 1$ be positive or negative according as the first of the three factors which is $\neq 1$ is a positive or a negative element of the subgroup A, B, $[A, B]$ respectively. Then the positive and the negative elements form semigroups with 1 as their only common element and which are inverse to one another; all the elements $\neq 1$ are either positive or negative. It remains only to prove that this partition into semigroups is normal invariant. Put $c' = a'b'\gamma'$, then $c'ab\gamma.c'^{-1} = ab\gamma[a^{-1}, b'][a', b]$ and it is therefore positive or negative according as $ab\gamma$ is positive or negative.

By an inner automorphism $A\text{o}B$ is transformed into $(cAc^{-1})\text{o}(cBc^{-1})$, but for $c = ab\gamma$, there is $cAc^{-1} = bAb^{-1}$ and $cBc^{-1} = aBa^{-1}$. Hence $A\text{o}B = bAb^{-1}\text{o}\, aBa^{-1}$, where a is an arbitrary element of A, and b an arbitrary element of B. Thus bAb^{-1} and aBa^{-1} are arbitrary conjugates of A and of B respectively. Hence:

THEOREM 4. *In an S-product* $A\text{o}B$, *the factors A and B can be replaced independently by conjugate groups.*

The corresponding theorem holds for direct products but not for free products. Moreover $A_1 = b_1{}^{-1}Ab_1 = A[a, b_1]$, where a runs over the corresponding elements in A. The product of n subgroups of A which are conjugate to A in $A\text{o}B$ is therefore $A_1A_2\ldots A_n$ and consists of the elements $a_1\ldots a_n[a_1, b_1]\ldots[a_n, b_n]$, where $b_1,\ldots, b_n$ are fixed elements and $a_1,\ldots, a_n$ are running independently over A. Putting $a_1\ldots a_m = a$, $a_1a^{-1} = a_1'$, we get $a.[a, b_1]$ $[a_1', b_1][a_2, b_2]\ldots[a_n, b_n]$, where $a_1'a_2\ldots a_n = 1$. The products $[a_2', b_2]$ $[a_2, b_2]\ldots[a_n, b_n]$ form a group which has no element $\neq 1$ in common with A_1. Hence—omitting the dash of a'_1—$A_1 \,.\, A_n = A_1\times[a_1, b_1]\ldots[a_n, b_n]$, where $a_1\ldots a_n = 1$. On the right side, A_1 can be replaced by any other A_k of the product on the left.

In Note II the mappings $\{c\}$ were considered which map $x \rightarrow (c, x)$ and it was shown that they are homomorphisms of a group G if and only if $((x_1, c), x_2) = 1$

for every pair of elements x_1 and x_2 of G. Put $G = A\circ B$, $c = a\,b\gamma$, $c_1 = a_1 b_1 \gamma_1$, $c_2 = a_2 b_2 \gamma_2$, then $((c_1, c), c_2) = ((a_1, a), a_2)\ ((b_1, b) b_2)$. The right side is equal to 1 if and only if both the factors are equal to 1. Hence:

THEOREM 5. *$\{a\,b\gamma\}$ is a homomorphism of $A\circ B$ if and only if $\{a\}$ is a homomorphism of A and $\{b\}$ is a homomorphism of B.*

A GENERAL EXPRESSION FOR EULER'S ϕ-FUNCTION.

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[Received 10 March, 1944.]

In the present note we shall give a general expression for Euler's ϕ-function. Let $\mu_k(n)$ be defined as a generalized Möbius function* by

$$\mu_k(n) = \begin{cases} 0 \text{ if } n \text{ has a factor whose power is} > k, \\ (-1)^{l_1+\dots+l_s}\binom{k}{l_1}\dots\binom{k}{l_s} \text{ otherwise,} \end{cases} \tag{1}$$

where $n = p_1^{l_1}\dots p_s^{l_s}$ and $p_1, \dots, p_s$ are distinct prime factors of n. Let $(1-p^{-1})^{1-k}_{(l+1)}$ denote a partial sum of the first $(l+1)$ terms in the expansion of $(1-p^{-1})^{1-k}$, that is

$$(1-1/p)^{1-k}_{(l+1)} = 1 + \binom{k-1}{1}p^{-1} + \binom{k}{2}p^{-2} + \dots + \binom{k+l-2}{l}p^{-l},\ (p > 1). \tag{2}$$

Then there exists an arithmetical function $\sigma_k(n)$ which may be placed by

$$\sigma_k(n) = n \prod_{p \mid n} (1-1/p)^{1-k}_{(l+1)}, \tag{3}$$

the product being extended over all prime factors p of n and l being the corresponding power of p, such that

$$\phi(n) = \sum_{d \mid n} \mu_k(n/d)\sigma_k(d) = \sum_{d \mid n} \mu_k(d)\sigma_k(n/d) \tag{4}$$

* The original Möbius function, $\mu(n)$, is defined as follows: $\mu(1) = 1$, $\mu(n) = 0$ if n has a squared factor; $\mu(p_1 p_2 \dots p_r) = (-1)^r$, if all the primes $p_1, p_2, \dots, p_r$ are different.

holds for every integer $k \geqslant 1$.

Putting $k = 1$, we find $\sigma_1(n) = n$, $\mu_1(n) = \mu(n)$ and

$$\phi(n) = \sum_{d \mid n} \mu(n/d)d = n \sum_{d \mid n} \mu(d)/d = \prod_{p \mid n} (p^l - p^{l-1}). \qquad (5)$$

This is the original expression for $\phi(n)$. We may therefore call (4) a general expression or a representation concerning Euler's function $\phi(n)$.

For proof of (4) we shall use the combinatorial equality,

$$\sum_{x=0} (-1)^x \binom{k}{x} \binom{s-x+k-1}{k-1} = \begin{cases} 1 \text{ if } s = 0, \\ 0 \text{ if } s > 0. \end{cases} \qquad (6)$$

It is sufficient to prove it for $s \geqslant 1$. Let $k - x = y$, we have

$$\sum_{x=0}^{s} (-1)^x \binom{k}{x} \binom{s-x+k-1}{k-1} = (-1)^k \sum_{y=k-s}^{k} (-1)^y \binom{k}{y} \times \binom{y+s-1}{k-1}$$

$$= (-1)^k \sum_{y=k-s}^{k} \binom{k}{y} \left\{ \binom{y}{0} \binom{s-1}{k-1} + \ldots + \binom{y}{k-1} \binom{s-1}{0} \right\}.$$

By a well-known equality

$$\sum_{b=c}^{a} (-1)^{a-b} \binom{a}{b} \binom{b}{c} = \begin{cases} 1 & \text{for } a = c \\ 0 & \text{for } a \neq c, \end{cases}$$

the last expression is reduced to zero and (6) is proved.

Now, writing

$n = p_1^{l_1} \cdots p_s^{l_s} = \Pi p^l$, $d = p_1^{t_1} \cdots p_s^{t_s} = \Pi p^t$, $h = p_1^{j_1} \cdots p_s^{j_s} = \Pi p^j$ and using (3), (5), (1) and (6) we have

$$\sum_{d \mid n} \mu_k(n/d) \sigma_k(d) = \sum_{d \mid n} \mu_k(n/d) \prod_{p \mid n} \left\{ \sum_{j=0}^{j=t} \binom{t-j+k-2}{k-2} p^j \right\}$$

$$= \sum_{d|n} \mu_k(n/d) \prod_{p|n} \left[\sum_{j=0}^{j=t} \binom{t-j+k-1}{k-1} (p^j - p^{j-1}) \right]$$

(defining $p^{j-1} = 0$ if $j = 0$.)

$$= \sum_{d|n} \mu_k(n/d) \sum_{h|d} \left[\prod_{p|n} \binom{t-j+k-1}{k-1} (p^j - p^{j-1}) \right]$$

$$= \sum_{d|n} \mu_k(n/d) \sum_{h|d} \phi(h) \prod_{p|n} \binom{t-j+k-1}{k-1}$$

$$= \sum_{h|n} \phi(h) \left[\sum_{h|d,\, d|n} \mu_k(n/d) \prod_{p|n} \binom{t-j+k-1}{k-1} \right]$$

$$= \sum_{h|n} \phi(h) \prod_{p|n} \left[\sum_{j \leqslant t \leqslant l} (-1)^{l-t} \binom{k}{l-t} \binom{t-j+k-1}{k-1} \right]$$

$$= \sum_{h|n} \phi(h) \prod_{p|n} \left[\sum_{x=0}^{x=l-j} (-1)^x \binom{k}{x} \binom{l-j-x+k-1}{k-1} \right]$$

$$= \phi(h)|_{l=j} = \phi(n).$$

Therefore (4) follows.

Denote by $(1/p)_{\overline{(k-1)}}^{-l-1}$ the partial sum of the first $(k-1)$ terms in the expansion of $\{ 1-(p-1)/p \}^{-l-1}$. Then by induction it can be shown that

$$(1-1/p)_{(l+1)}^{1-k} = (1-1/p)^{1-k} \{ 1-(1/p)^{l+1}(1/p)_{\overline{(k-1)}}^{-l-1} \},\ p > 1. \quad (7)$$

Thus $\sigma_k(n)$ can be replaced by

$$\sigma_k(n) = n \prod_{p|n} (1-1/p)^{1-k} \left[1-(1/p)^{l+1}(1/p)_{\overline{(k-1)}}^{-l-1} \right]. \quad (8)$$

Evidently it is more convenient to use (8) when k is small.

Numerical example: Given $k=2$, $n=375$, we find

$$\sigma_2(d) = \sigma_2(p_1^{t_1} \dots p_s^{t_s}) = \left\{ \frac{p_1^{t_1+1}-1}{p_1-1} \right\} \dots \left\{ \frac{p_s^{t_s+1}-1}{p_s-1} \right\},$$

$\phi(375) = \mu_2(1)\sigma_2(375) + \mu_2(3)\sigma_2(125) + \mu_2(5)\sigma_2(75) + \mu_2(15)\sigma_2(25) + \mu_2(25)\sigma_2(15) + \mu_2(75)\sigma_2(5) + \mu_2(125)\sigma_2(3) + \mu_2(375)\sigma_2(1)$
$= 1.624 + (-2).156 + (-2).124 + 4.31 + 1.24 + (-2).6 + 0 + 0$
$= 200.$

Lastly we note that the factorization of an integer may not necessarily be put into a canonical form.

Let $q_1, \ldots, q_t$ *be a set of given factors by which the numbers* n *and* d *are defined to be*

$$n = q_1^{m_1} \ldots q_t^{m_t},\ d = q_1^{n_1} \ldots q_t^{n_t}\ (0 \leqslant n_i \leqslant m_i)$$

while the generalized Möbius function $\mu_k(n)$ *and the Arithemetic function* $\sigma_k(n)$ *are redefined to be*

$$\mu_k(n) = \begin{cases} 0 \text{ if } n \text{ has a factor where power} > k, \\ (-1)^{m_1 + \ldots + m_t} \binom{k}{m_1} \ldots \binom{k}{m_t} \text{ otherwise;} \end{cases} \tag{9}$$

$$\sigma_k(n) = n \prod_{q \mid n} (1 - 1/q)^{1-k}_{(m+1)}. \tag{10}$$

And let $q_i = p_1^{h_{i1}} \ldots p_s^{h_{is}}$, $(s \geqslant t,\ i = 1, \ldots, t)$, $p_1, \ldots, p_s$ *being different primes. Then the necessary and sufficient condition that* (4) *holds for all* k *is that the rank of the matrix* $\|h_{ij}\|$ $(i = 1, \ldots, t;\ j = 1, \ldots, s)$ *is equal to* t.

ON SINGULAR FUNCTIONS*

by

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[Received 21 May, 1944.]

1. The object of this paper is to discuss some properties of functions of a real variable possessing derivatives of all orders. If $f(x)$ possesses derivatives of all orders in $(x_0 - \delta, x_0 + \delta)$, we can associate with it the Taylor series

$$f(x_0) + (x - x_0) f'(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \ldots \tag{1}$$

Now it is known that even if (1) converges for all x in $|x - x_0| \leqslant \delta$, it is not necessary that the sum should be $f(x)$. The failure of the uniqueness of Taylor series is in curious contrast to the corresponding property of the Fourier series of a function $f(x)$ integrable $(-\pi, \pi)$.

* Communicated to the 13th conference of the *Indian Mathematical Society*, at Annamalainagar (December, 1943).

It is known that if $f(x)$ and $g(x)$ have the same Fourier series, then $f(x)=g(x)$ in $(-\pi, \pi)$ except possibly over a null set even if the Fourier series does not converge at any point in $(-\pi, \pi)$.

2. When a, b are finite, (a, b) shall always indicate a closed interval. $C(a, b)$ denotes the class of all functions $f(x)$ defined in (a, b) and possessing derivatives of all orders in (a, b). $f(x)$ of $C(a, b)$ is said to be *singular in* (a, b) if (i) $f(x) \not\equiv 0$ in (a, b) and (ii) there is a point x_0 in (a, b) such that $f^{(n)}(x_0)=0$ for $n=0, 1, \ldots$, e.g. $f(x)=\exp\left\{1/(x-x_0)^2\right\}$, $x \neq x_0$; $f(x_0)=0$. $f(x)$ is *essentially singular in* (a, b) if for some n_0, $f^{(n_0)}(x)$ is singular in (a, b). $f(x)$ of $C(a, b)$ is said to be *singular at* x_0 in (a, b) if (i) $f(x) \not\equiv 0$ in any neighbourhood of x_0 and (ii) $f^{(n)}(x_0)=0$, $n=0, 1, 2, \ldots$

If for some n_0, $f^{(n_0)}(x)$ is singular at x_0, then $f(x)$ is said to be essentially singular at x_0. The point x_0 is called a singular point or an essentially singular point of $f(x)$ as the case may be. A function $f(x)$ of $C(a, b)$ is said to be *analytic at* x_0 of (a, b) if $f(x)=$ the Taylor series (1) in some neighbourhood of x_0. $f(x)$ is said to be *analytic in* (a, b) *if it is analytic at every point of* (a, b).

THEOREM 1. *A set of necessary and sufficient conditions that $f(x)$ belonging to $C(x_0, x_0+\delta)$ should be singular at x_0 is*

(i) $f(x) \not\equiv 0$ *in any interval* (x_0, x_0+h), $0<h\leqslant\delta$;

(ii) $f(x_0+h)/h^n \to 0$ *as* $h \to +0$ *for every integer* $n \geqslant 0$.

PROOF: The conditions are necessary. Obviously (i) follows from the definition of a singular point.

To prove (ii), since $f^{(n)}(x_0)=0$ for $n=0, 1, 2, \ldots$, we have for any given n,

$$f(x_0+h)=\frac{h^n}{n!}f^{(n)}(x_0+\theta h),\ 0<\theta<1$$

so that

$$\frac{f(x_0+h)}{h^n} \to \frac{f^{(n)}(x_0)}{n!}=0.$$

So the conditions are necessary. To prove that they are sufficient we see by (ii) and the continuity of $f(x)$ in $(x_0, x_0+\delta)$ that $f(x_0)=0$. Suppose now that we have proved that $f^{(n)}(x_0)=0$ for $n=0, 1, \ldots, m-1$. By Taylor's theorem

$$f(x_0+h)=\frac{h^m}{m!}\left\{f^{(m)}(x_0)+\frac{h}{m+1}f^{m+1}(x_0+\theta h)\right\}, 0<\theta<1,$$

so that

$$f^{(m)}(x_0)/m! = \lim_{h\to+0} f(x_0+h)/h^m = 0$$

by (ii). So it follows by induction that $f^{(n)}(x_0)=0$ for $n=0, 1, 2, \ldots$. This along with (i) shows that x_0 is a singular point of $f(x)$.

2.1. A similar theorem can be easily stated for an interval $(x_0-\delta, x_0)$ or $(x_0-\delta, x_0+\delta)$. It must be noted that the condition that $f(x)$ belongs to $C(x_0, x_0+\delta)$ is essential. It is not enough if $f(x)$ has derivatives of all orders in $x_0<x\leqslant x_0+\delta$. This may be seen by considering $g(x)$ defined by

$$g(x_0)=0$$

$$g(x)=\exp\left[-1/(x-x_0)^2\right]\sin\left\{\exp\left[2/(x-x_0)^2\right]\right\},\ x\neq x_0$$

in $x_0<x\leqslant x_0+\delta$. Obviously

$$g(x_0+h)/h^n\to 0 \text{ as } h\to+0 \text{ for } n=0, 1, 2, \ldots$$

But $g''(x_0)$ does not exist.

2.2. THEOREM 2. *A necessary and sufficient condition that $f(x)$ belonging to $C(a, b)$ should be singular in (a, b) is that it should have an infinite number of zeros in (a, b) while $f(x)\not\equiv 0$ in (a, b).**

PROOF. The given conditions are obviously necessary by definition. To prove that they are sufficient, the result is obvious when there is an x_0 where $f^{(n)}(x_0)=0$ for $n=0, 1, 2, \ldots$ If not, there will be a sequence $\{x_n\}$

* If, for some x_0 in (a, b), $f^{(n)}(x_0)=0$ for $n=0, 1, \ldots$ then x_0 is a zero of infinite order for $f(x)$, and $f(x)$ is regarded as having an infinite number of zeros even if $f(x)\neq 0$ at any other point.

in (a, b) such that $f(x_n) = 0$. Let α be a limit point of $\{x_n\}$. Then we shall prove that $f^{(n)}(\alpha) = 0$ for $n = 0, 1, 2, \ldots$ If not, there will be a least integer $m \geqslant 0$ such that $f^{(m)}(\alpha) \neq 0$. By Taylor's theorem,

$$f(\alpha+h) = \frac{h^m}{m!}\left\{f^{(m)}(\alpha) + \frac{h}{1+m} f^{m+1}(\alpha+\theta h)\right\}$$

Since $f^{(m)}(\alpha) \neq 0$, we can find a $\delta > 0$ such that $f(\alpha+h) \neq 0$ for $|h| \leqslant \delta$. But α is a limit point of $\{x_n\}$ so that we can find an infinite number of values of h in $|h| \leqslant \delta$ such that $f(\alpha+h) = 0$. This gives a contradiction. So $f^{(n)}(\alpha) = 0$ for $n = 0, 1, 2, \ldots$ Hence $f(x)$ is singular in (a, b).

2.3. THEOREM 3. *A function $f(x)$ of $C(a, b)$ is singular in (a, b) if and only if it has a singular point in (a, b).*

PROOF. If $f(x)$ has a singular point in (a, b) it is obviously singular in (a, b). Suppose now that $f(x)$ is singular in (a, b). Then there is a point x_0 in (a, b) such that $f^{(n)}(x_0) = 0$. If $f(x) \not\equiv 0$ in any neighbourhood of x_0, then x_0 is a singular point of $f(x)$. If not there is a neighbourhood of x_0 in which $f(x) \equiv 0$. Let α be the lower bound of numbers δ such that $f(x) \equiv 0$ in $(x_0 - \delta, x_0)$ and β the upper bound of numbers δ such that $f(x) \equiv 0$ in $(x_0, x_0 + \delta)$. Now it cannot happen that $\alpha = a$ and $\beta = b$ simultaneously in which case $f(x) \equiv 0$ in (a, b) since $f(x)$ is continuous in (a, b). So at least one of α and β must be inside (a, b). Let α be inside (a, b). Then since $f(x) \equiv 0$ in $\alpha < x \leqslant \alpha + \delta$ for some $\delta > 0$, we have $f^{(n)}(\alpha) = 0$ for $n = 0, 1, 2, \ldots$ Also $f(x) \not\equiv 0$ in any interval of the type $(\alpha - \delta, \alpha)$ by the definition of α. Hence α is a singular point of $f(x)$. A similar proof holds if β is inside (a, b).

2.4. THEOREM 4. *The set S of singular points of $f(x)$ of $C(a, b)$ is either empty or a closed non-dense set.*

PROOF. Suppose that S is non-empty. If S contains only a finite number of points then it is closed and non-dense. Suppose S is an infinite set. Let α be a limit point of S. Since $f^{(n)}(x)$ is continuous in (a, b), it follows

from the definition of S that $f^{(n)}(\alpha)=0$ for $n=0, 1, 2, \ldots$ Let $(\alpha-\delta, \alpha+\delta)$ be any neighbourhood of α. It contains a point β of S and so an interval I containing β inside the neighbourhood. Since $f(x) \not\equiv 0$ in I, it follows that $f(x) \not\equiv 0$ in $(\alpha-\delta, \alpha+\delta)$. So α is a singular point of $f(x)$ and so belongs to S. If S is not non-dense in (a, b), it will be everywhere dense in some sub-interval I of (a, b) and since S is closed, I belongs to S and so $f(x) \equiv 0$ in I so that no interior point of I can be a singular point of $f(x)$ and so cannot be a point of S. So this gives a contradiction. Therefore S is non-dense.

3. Theorems 5 and 7 contain the main results of this paper.

THEOREM 5. *Let $f(x)$ belong to $C(a, b)$. If $f^{(n)}(x) \geqslant 0$ in (a, b) for every integer $n \geqslant 0$ then $f(x)$ is analytic in $(a, b-\delta)$ where $0 < \delta < b-a$.*

3·1. The proof is based on the following known* lemma the proof of which is added for completeness. The proof of sufficiency appears to be new.

LEMMA 1. *Let $f(x)$ be a function of $C(a, b)$. Let $M_n = \max\limits_{a \leqslant x \leqslant b} |f^{(n)}(x)|$. Then a necessary and sufficient condition that $f(x)$ should be analytic in (a, b) is that*

$$\overline{\lim_{n=\infty}} \{ M_n/n! \}^{\frac{1}{n}} < \infty.$$

PROOF. The condition is necessary. By the definition of analyticity we can find a closed region Γ in the complex plane containing (a, b) in its interior and a function $g(z)$ regular in Γ such that $g(x)=f(x)$ in (a, b). Let δ be the distance of (a, b) from the boundary of Γ. Then for any x in (a, b)

$$\frac{f^{(n)}(x)}{n!} = \frac{g^{(n)}(x)}{n!} = \frac{1}{2\pi i}\int_{|z-x|=\delta} \frac{g(z)\,dz}{(z-x)^{n+1}}. \tag{2}$$

* Mandelbrojt, "*Series de Fourier et Classes Quasi-analytiques*" (Borel Tracts, 1935), p. 48.

If $M = \max |g(z)|$ for z in Γ we have by (4) $\frac{M_n}{n!} \leqslant \frac{M}{\delta^n}$ so that $\overline{\lim_{n\to\infty}} \left\{ \frac{M_n}{n!} \right\}^{1/n} \leqslant \frac{1}{\delta} < \infty$. So the condition is necessary. To prove that the condition is sufficient let

$$\rho = \overline{\lim_{n\to\infty}} \{ M_n/n! \}^{1/n},$$

so that for a suitable n_0

$$[M_n/n!]^{1/n} \leqslant \rho + 1, \text{ for } n \geqslant n_0. \qquad (3)$$

Let $n > n_0$. By Taylor's theorem if x and x_0 are in (a, b) we have

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n, \qquad (4)$$

where

$$|R_n| = \left| \frac{f^{(n+1)}(x_0 + \theta h)}{(n+1)!}(x - x_0)^{n+1} \right|$$
$$\leqslant |(\rho + 1)(x - x_0)|^{n+1}, \qquad (5)$$

by (3). Hence if $|x - x_0| < 1/(\rho + 1)$, $R_n \to 0$ as $n \to \infty$ by (5), and it follows by (4) that as $f(x)$ is analytic at x_0, x_0 being any point in (a, b), $f(x)$ is analytic in (a, b). This proves Lemma 1.

3.2. Proof of Theorem 5. Let x be any point in (a, b). Then by Taylor's theorem

$$f(b) = f(x) + (b - x)f'(x) + \ldots + \frac{(b-x)^n}{n!} f^{(n)}(x)$$
$$+ \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x'), \qquad (6)$$

where $x < x' < b$. Now by hypothesis each term in (6) is non-negative. Hence

$$\frac{(b-x)^n}{n!} f^{(n)}(x) \leqslant f(b) \text{ for } a \leqslant x \leqslant b.$$

If x is in $(a, b - \delta)$ and $M_n = \max_{a \leqslant x \leqslant b-\delta} |f^{(n)}(x)|$, we see from the last equation that

$$M_n/n! \leqslant f(b)/\delta^n,$$

that is, $\overline{\lim_{n\to\infty}}\{M_n/n!\}^{1/n} \leqslant 1/\delta < \infty$. So by Lemma 1 $f(x)$ is analytic in $(a, b-\delta)$.

3·3. THEOREM. 6. *Let $f(x)$ be a function of $C(a, b)$. Let $(-1)^n f^{(n)}(x) \geqslant 0$ for $a \leqslant x \leqslant b$ and for each $n = 0, 1, 2, \ldots$ Then $f(x)$ is analytic in $(a+\delta, b)$ where $0 < \delta < b-a$.*

PROOF. Starting from the equation

$$f(a) = f(x) + (a-x)f'(x) + \ldots + \frac{(a-x)^{n+1}}{(n+1)!} f^{(n+1)}(x') \quad \text{where}$$

$a < x' < x$, the proof is similar to that of Theorem 5.

3·4. It is obvious that $\geqslant 0$ can be replaced by $\leqslant 0$ in the hypotheses of Theorems 5 and 6. It may be also noted that the theorems are *best possible* in the sense that we cannot assert that $f(x)$ is analytic in (a, b). This is easily seen by considering

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n^2}}{2^n}$$

in $(0 \leqslant x \leqslant 1)$. Also by putting $x = a$ in (6) we see that

$$\sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

has a radius of convergence at least equal to $(b-a)$. So that under the hypothesis of Theorem 5, $f(x)$ can be *extended analytically over the interval* $(2a-b, a)$.

3·5. THEOREM 7. *Let $f(x)$ belong to $C(a, b)$. Let $\{n_i\}$ be a steadily increasing sequence of positive integers tending to infinity. Let $f^{(n_i)}(x)$ not change sign in (a, b) for each n_i [even though the sign of $f^{(n_i)}(x)$ is different for different n_i]. Then $f(x)$ cannot be singular in (a, b).*

PROOF. If $f(x)$ is singular in (a, b) then there is at least one singular point in (a, b) by Theorem 3. Let $x = a$ be such a singular point. Then

$$f^{(k)}(x) = \frac{1}{(n-k-1)!} \int_a^x (x-t)^{n-k-1} f^{(n)}(t)\,dt, \qquad (7)$$

$0 \leqslant k < n$. Putting $n = n_i$ and using the hypothesis, we see that $f^{(k)}(x) \geqslant 0$ (or $\leqslant 0$) in (a, b) for all k such that $0 \leqslant k \leqslant n_i$

Since $n_i \to \infty$ as $i \to \infty$, it follows that $f^{(n)}(x) \geqslant 0$ (or $\leqslant 0$) in (a, b) for all $k = 0, 1, 2, \ldots$ So by Theorem 5 $f(x)$ is analytic in $(a, b-\delta)$ and since $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \ldots$ we see that $f(x) \equiv 0$ in $(a, b-\delta)$ so that $x = a$ cannot be a singular point of $f(x)$. Similarly by using Theorem 6 we can prove that $x = b$ cannot be a singular point. If x_0 is any point in (a, b) we can consider (a, x_0), and (x_0, b) separately and conclude that x_0 cannot be a singular point of $f(x)$. So $f(x)$ cannot be singular in (a, b).

3.6. Theorem 8. *Let $f(x)$ be singular in (a, b), x_0 a singular point of $f(x)$. Given any neighbourhood of x_0, we can find an n_0 such that each $f^{(n)}(x)$, $n \geqslant n_0$, changes sign at least once in this neighbourhood.*

Proof. If the theorem were not true, $f(x)$ would satisfy the condition of Theorem 7 in some neighbourhood of x_0 and for some sequence $\{n_i\}$ so that x_0 cannot be a singular point according to the hypothesis.

3.7. Let $f(x)$ belong to $C(a, b)$. Let μ_n denote the number of changes of sign of $f^{(n)}(x)$ in (a, b). Then it is easily seen that $\mu_n \leqslant \mu_{n+1} + 1$ provided μ_{n+1} is finite. From this it follows that if, for some n, μ_n is infinite, then $\mu_{n+1}, \mu_{n+2}, \ldots$ are all infinite and in this case $f^{(n)}(x)$ is singular by Theorem 2, that is, $f(x)$ is essentially singular. If $f(x)$ is singular or essentially singular, Theorem 8 states that $\mu_n \geqslant 1$ after a certain stage. It appears quite probable that $\mu_n \to \infty$ when $f(x)$ is singular or essentially singular in (a, b) but I have not been able to prove or disprove it.

3.8. Let Z_n denote the set of distinct zeros of $f^{(n)}(x)$ in (a, b) without regard to multiplicity. We prove

Theorem 9. *Let $f(x)$ be a function singular in (a, b) and let x_0 be a singular point of $f(x)$ in (a, b). Let $\{n_i\}$ be any sequence of positive integers tending to infinity. Let $Z\{n_i\} = Z_{n_1} + Z_{n_2} + \ldots$. Then x_0 is a limit point of $Z\{n_{ni}\}$.*

PROOF. If x_0 is not a limit point of $\mathcal{Z}\{n_i\}$ we can find a $\delta > 0$ such that $(x_0-\delta, x_0+\delta)$ contains no point of $\mathcal{Z}\{n_i\}$ except x_0 itself. Therefore the sequence $\{f^{(n_i)}(x)\}$ satisfies the conditions of Theorem 7 in $(x_0-\delta, x_0)$ and $(x_0, x_0+\delta)$ so x_0 cannot be a singular point of $f(x)$ in either of these intervals—that is, x_0 cannot be a singular point of $f(x)$ in (a, b) contrary to hypothesis.

3·9. The converse of Theorem 9 is not true. We cannot assert that any limit point of $\mathcal{Z}\{n\}$ or $\mathcal{Z}\{n_i\}$ will always be a singular point. For instance, consider $f(x) = \sin x$ in $(0, 2\pi)$. The set $\mathcal{Z}\{n\}$ is composed of the points $(0, \pi/2, \pi, 3\pi/2)$ each repeated infinitely so that each of these points is a limit point of $\mathcal{Z}\{n\}$. But none of these is a singular point of $f(x)$.

4. *Quasi-analytic classes.* A class of functions each of which belongs to $C\,(a, b)$ is said to be *quasi-analytic in* (a, b) if it does not include a singular function. Otherwise the class is called *a singular class.*

If the class of functions is a linear class or module, that is, if $c_1 f_1(x) + c_2 f_2(x)$ belongs to the class when $f_1(x)$ and $f_2(x)$ belong to the class, for all real numbers c_1 and c_2, the above definition can be restated as follows:—

A linear class of functions belonging to $C(a, b)$ is said to be quasi-analytic in (a, b) if, $f_1(x)$ and $f_2(x)$ being two functions of the class, the relations $f_1^{(n)}(x_0) = f_2^{(n)}(x_0)$ for some x_0 in (a, b) and for $n = 0, 1, 2, \ldots$, imply that $f_1(x) \equiv f_2(x)$ in (a, b). In other words, a function $f(x)$ of the class is entirely determined by the values of $f^{(n)}(x)$ at any one point $x = x_0$ of (a, b).

4.1. The most widely studied class of functions* is defined as follows. Let $\{A_n\}$ be any sequence of positive numbers. Let $C\{A_n\}$ denote the class of all functions $f(x)$ of $C\,(a, b)$ such that

* T. Carleman, *Fonctions Quasi-analytiques* (Borel Series, 1926).

$$\overline{\lim_{n\to\infty}}[M_n/A_n]^{1/n} < \infty, \tag{8}$$

where $M_n = \max_{a \leqslant x \leqslant b} |f^{(n)}(x)|$. The class thus defined is easily seen to be linear. The relation (8) implies and is implied by the statement that for each function $f(x)$ of $C\{A_n\}$ there is a $k > 0$ such that $|f^{(n)}(x)| \leqslant k^n A_n$, $a \leqslant x \leqslant b$. By Lemma 1, the class $C\{n!\}$ is identical with the class of analytic functions. The following general theorem has been proved by Carleman.*

THEOREM 10. *Let $\alpha_n = A_n^{1/n}$ and α'_n = the lower bound of α_p for $p \geqslant n$. A necessary and sufficient condition that the class $C\{A_n\}$ should be quasi-analytic is that the integral*

$$\int_1^\infty \log\left(\sum_{\nu=0}^{\infty} \frac{r^{2\nu}}{A_\nu^2}\right)\frac{dr}{r^2} \tag{9}$$

should diverge. This condition is equivalent to the statement that $\Sigma 1/\alpha'_n$ should diverge.

4.2. From Theorem 10 it follows that a class $C\{A_n\}$ of functions of $C(a, b)$ includes at least one singular function when (9) converges. We have already seen that the set of singular points of a function singular in (a, b) is a closed non-dense set. The question now arises whether a singular class $C\{A_n\}$, for which (9) converges, includes functions with more than one singular point and in general, a function with any given closed non-dense set as singular points. The answer to this question is in the affirmative. We shall prove

THEOREM 11. *Let S be any closed non-dense set in (a, b). Let $\{A_n\}$ be a positive sequence such that (9) converges, so that the class $C\{A_n\}$ of functions belonging to $C(a, b)$ is singular. Then we can find $f(x)$ of $C\{A_n\}$ such that S is the set of singular points of $f(x)$ in (a, b).*

4.3. The proof is based on the following lemmas.

* T. Carleman, *l.c.* pp. 60-4.

LEMMA 2.* *Let* $\{A_n\}$ *be a positive sequence such that* (9) *converges. Then there exists* $f(x)$ *of* $C(0, \infty)$ *which is singular in* $(0, \infty)$ *such that*

(i) $f^{(n)}(0) = 0,\ n = 0, 1, 2, \ldots$

(ii) $|f^{n}(x)| \leqslant A_n,\ 0 \leqslant x < \infty,\ n = 0, 1, 2, \ldots$.

LEMMA 3. *Let* $\{A_n\}$ *be such that* (9) *converges. Then we can find* $f(x)$ *of* $C(-\infty, \infty)$ *and belonging to* $C\{A_n\}$ *such that* $x = 0$ *is the only singular point of* $f(x)$ *in* $(-\infty, \infty)$.

PROOF. By Lemma 2, there is a singular function $g(x)$ of $C(0, \infty)$, such that

$$g^{(n)}(0) = 0,\ |g^{(n)}(x)| \leqslant A_n,\ n = 0, 1, 2, \ldots$$

Let (α, β) be one of the complementary intervals of the set of singular points of $g(x)$ in $(0, \infty)$. If we put $h(x) = g(\overline{x-\alpha}/\beta-\alpha)$, then $h(x)$ is singular in $0 \leqslant x \leqslant 1$ and has no singular points in $(0, 1)$ other than $x = 0$ and $x = 1$; $|h^{(n)}(x)| \leqslant A_n/(\beta-\alpha)^n,\ 0 \leqslant x \leqslant 1,\ n = 0, 1, 2, \ldots$ If we put $x = t/(1+t)$, then $p(t) = h(t/\overline{1+t})$ is singular in $(0, \infty)$ and $t = 0$ is its only singular point; and

$$|p^{(n)}(t)| \leqslant [2e/(\beta-\alpha)]^n A_n \dagger.$$

If we now put $f(x) = p(x)$ in $0 \leqslant x < \infty$ and $f(-x) = f(x)$ we get $f(x)$ of the required type. This proves Lemma 3

4.5. LEMMA 4. *Let* $\{A_n\}$ *be such that* (9) *converges. Let* (a, b) *be such that* $b-a \leqslant k > 0$ *Then there is* $u(x)$ *of* $C(a, b)$ *belonging to* $C\{A_n\}$ *such that* $u(x)$ *is singular in* (a, b) *and* $x = a$ *and* $x = b$ *are its only singular points; and* $|u^{(n)}(x)| \leqslant (k\lambda)^n A_n,\ n = 0, 1, 2, \ldots,$ *where* λ *is independent of the interval* (a, b).

PROOF. Let $f(x)$ be the function defined in Lemma 3 with reference to $\{A_n\}$. Let $M_n = \max\limits_{-\infty<x<\infty} |f^{(n)}(x)|$. Then we have if $\lambda_1 = 2e/(\beta-\alpha)$,

$$M_n \leqslant \lambda_1^n A_n. \tag{10}$$

* See T. Carleman, *l.c.* p. 62.

† See Carleman, *l.c.* pp. 22-3.

Since $f^{(n)}(0) = 0$ we have, if $0 \leqslant p < n$,

$$f^{(p)}(x) = \frac{1}{(n-p-1)!}\int_0^x (x-t)^{n-p-1} f^{(n)}(t)\, dt$$

so that
$$M_p \leqslant \frac{M_n}{(n-p)!}, \quad 0 \leqslant p < n. \tag{11}$$

Now let

$$u(x) = f[(x-a)\ (b-x)], \quad a \leqslant x \leqslant b.$$

Then it is seen that $u(x)$ belongs to $C(a, b)$ and $x = a$ and $x = b$ are its only two singular points. To prove the second part, we use the result*: Let $|f^{(n)}(x)| \leqslant B_n$ in $a \leqslant x \leqslant b$ and $|\phi^{(n)}(t)| \leqslant C_n$ in $t_1 \leqslant t \leqslant t_2$, where $a \leqslant \phi(t) \leqslant b$ in $t_1 \leqslant t \leqslant t_2$. Let $h(t) = f\{\phi(t)\}$ and

$$f_n(x) = \sum_{\nu=0}^{n} B_\nu x^\nu/\nu!;\ \phi_n(x) = \sum_{\nu=0}^{n} C_\nu x^\nu/\nu!.$$

Then

$$|h^{(n)}(t)| \leqslant \frac{d^n}{dt^n}\Big[f_n\{\phi_n(t)\}\Big]_{t=0}, \quad t_1 \leqslant t \leqslant t_2. \tag{12}$$

Taking $\phi(x) = (x-a)\ (b-x)$ we have $|\phi'(x)| = |a+b-2x|$ so that $|\phi'(x)| \leqslant |b-a| \leqslant k$ in (a, b) while $|\phi''(x)| = 2$ and $\phi^{(n)}(x) = 0$ for $n \geqslant 3$. Also $-\infty < \phi(x) < \infty$ when $a \leqslant x \leqslant b$. So using (12) we get for $a \leqslant x \leqslant b$,

$$|u^{(n)}(x)| \leqslant \frac{d^n}{dx^n}\left\{\sum_{\nu=0}^{n} \frac{M_\nu}{\nu!}\,[kx+x^2]^\nu\right\}_{x=0}$$

By (11), $|u^{(n)}(x)| \leqslant \dfrac{M_n}{n!}\dfrac{d^n}{dx^n}\left\{\sum_{\nu=0}^{n}\binom{n}{\nu}(kx+x^2)^\nu\right\}_{x=0}$

$$= \frac{M_n}{n!}\left\{\frac{d^n}{dx^n}(1+kx+x^2)^n\right\}_{x=0} \leqslant \frac{M_n}{n!}\frac{d^n}{dx^n}\left\{\frac{1}{(1-kx)^n}\right\}_{x=0}$$

because $k > 1$. So*

$$|u^{(n)}(x)| \leqslant \frac{M_n}{n!}\ k^n \frac{(2n)!}{2(n!)} \leqslant (2ek)^n M_n, \quad a \leqslant x \leqslant b.$$

So by (10),

$$|u^{(n)}(x)| \leqslant (2ek\lambda_1)^n A_n = (k\lambda)^n A_n,$$

where $\lambda = 2e\lambda_1 = 4e^2/(\beta-\alpha)$ is independent of (a, b).

* T. Carleman, *l.c.*, p. 22.

4 6. PROOF OF THEOREM 11. If I is any interval we shall denote by $u(x, I)$ the function defined in Lemma 4 for the interval I. Let now (a, b) be any finite interval and let $\{A_n\}$ be a sequence for which (9) converges. Let S be a closed non-dense set in (a, b). Let $\bar{S}$ be the complementary set of intervals in (a, b). We define $f(x)$ as $f(x) = 0$ for any x belonging to S, $f(x) = u(x, I)$ for an interval I of $\bar{S}$. This function has the required property in (a, b). First, we shall prove that $f(x)$ belongs to $C(a, b)$. To prove this it is enough to show that $f'(x)$ exists in (a, b) and vanishes for x on S. For; then we can repeat the argument with $f'(x)$ and show that $f''(x)$ exists in (a, b) and vanishes on S and so on. Now $f'(x)$ exists at every point of $\bar{S}$ as well as at any isolated point of S where it vanishes. So let α be a limit point of S. We shall show that $f'(\alpha)$ exists and is zero. Let

$$g(h) = \frac{f(\alpha+h)-f(\alpha)}{h}.$$

If $\alpha+h$ belongs to S then $g(h) = 0$ since $f(\alpha+h) = f(\alpha) = 0$. If $\alpha+h$ belongs to $\bar{S}$, let I be the interval of $\bar{S}$ in which it lies and let $\alpha+\xi$ be that end point of I lying between α and $\alpha+h$. Since $f(\alpha+\xi) = 0$, we have

$$g(h) = \frac{f(\alpha+h)-f(\alpha+\xi)}{h} = \frac{u(\alpha+h, I)-u(\alpha+\xi, I)}{h}$$

$$= \frac{h-\xi}{h} u'(\alpha+\xi+\theta\overline{h-\xi}, I), \; 0 < \theta < 1.$$

Since $u'(\alpha+\xi, I) = 0$ we have

$$u'(\alpha+\xi+\theta\overline{h-\xi}, I) = u'(\alpha+\xi+\theta\overline{h-\xi}, I) - u'(\alpha+\xi, I)$$
$$= \theta.\overline{h-\xi}\, u''(\alpha+\xi+\theta'\theta\overline{h-\xi}, I), \; 0 < \theta' < 1.$$

Combining these, we get

$$|g(h)| \leqslant |(h-\xi)^2/h|\theta.|u''(\alpha+\xi+\theta\theta'\overline{h-\xi}, I)|.$$

Since $|u''(x, I)| \leqslant \{(b-a)\lambda\}^2 A_2$ [because the length of $I \leqslant b-a$] and $|h-\xi| \leqslant 2|h|$ we get

$$|g(h)| \leqslant 4\,|h|\,\{(b-a)\lambda\}^2 A_2, \tag{13}$$

when $\alpha+h$ is in S. Also (13) is obviously true when $\alpha+h$ is in S. Hence $g(h)\to 0$ as $h\to 0$ which shows that $f'(\alpha)=0$. This proves that $f(x)$ belongs to $C(a, b)$ and has S for its set of singular points. We have next to show that $f(x)$ belongs to $C\{A_n\}$. If x is a point of an interval I of $\bar{S}$ we have

$$|f^{(n)}(x)| = |u^{(n)}(x, I)| \leqslant (k\lambda)^n A_n, \tag{14}$$

since the length of I is $\leqslant b-a$ so that we can take $k=\max(1, b-a)$. If x is a point of S, then $f^{(n)}(x)=0$. Hence (14) is true in (a, b). So $f(x)$ belongs to $C\{A_n\}$. This proves Theorem 11.

5. I conclude with a few remarks on the convergence of the sequence $\{f^{(n)}(x)\}$ of the derivatives of $f(x)$ of $C(a, b)$. We shall first prove

THEOREM 12. *Let $f(x)$ be a function of $C(a, b)$ analytic in (a, b), that is, belongs to the class $C\{n!\}$. Let $f^{(n)}(x)\to l$ as $n\to\infty$ for some point x_0 of (a, b). Then $f^{(n)}(x)\to le^{x-x_0}$ uniformly for x in (a, b).*

PROOF. We shall prove that $f^{(n)}(x)\to le^{x-x_0}$ uniformly in some neighbourhood $(x_0-\delta/2, x_0+\delta/2)$ of x_0 where δ is the lower bound of the radii of convergence of the expansions of $f(x)$ about the several points of (a, b). We have, in $(x_0-\delta/2, x_0+\delta/2)$,

$$f^{(n)}(x) = \sum_{p=0}^{\infty} \frac{f^{(n+p)}(x_0)}{p!}(x-x_0)^p.$$

Now given ε, we can find n_0 such that $|f^{(n)}(x_0)-l| \leqslant \varepsilon$ for $n \geqslant n_0$. If $n \geqslant n_0$, we have

$$f^{(n)}(x)-le^{x-x_0} = \sum_{p=0}^{\infty} \{f^{(n+p)}(x_0)-l\}(x-x_0)^p/p!$$

so that $\quad |f^{(x)}(x)-le^{x-x_0}| \leqslant \varepsilon e^{\delta/2}$ if $n \geqslant n_0$

and $|x-x_0| \leqslant \delta/2$. The above shows that $f^{(n)}(x)\to le^{x-x_0}$ uniformly in $(x_0-\delta/2, x_0+\delta/2)$. We can now argue with

$x = x_0 \pm \delta/2$ and continue the process a finite number of times to prove the result stated.

5·1. The above theorem raises the general question of the convergence of the sequence $\{ f^{(n)}(x) \}$ of derivatives of $f(x)$ of $C(a, b)$. Theorem 12 states that if $f(x)$ is analytic and if $\{ f^{(n)}(x) \}$ converges to l at one point x_0 of (a, b), then $\{ f^{(n)}(x) \}$ converges at every point and the limit function is le^{x-x_0}. Very probably this result is true when $f(x)$ belongs to a quasi-analytic class $C\{ A_n \}$. In general, it is not true that the convergence of $\{ f^{(n)}(x) \}$ at a point implies its convergence in an interval round the point. For instance, if $f(x) = \exp(-1/x^2), x \neq 0, f(0) = 0$, the function belongs to $C(-\infty, \infty)$ while $f^{(n)}(0) = 0$ so that $\{ f^{(n)}(x) \}$ converges at $x = 0$. But a little calculation shows that $\{ f^{(n)}(1) \}$ cannot converge.

5·2. Even if we assume that the sequence $\{ f^{(n)}(x) \}$ of derivatives of $f(x)$ of $C(a, b)$ converges at every point of (a, b), t does not appearr possible to state that the limit is $f(a)e^{x-a}$. If it converges uniformly or even boundedly in (a, b) it follows from Theorem 12 that the limit is $f(a)e^{x-a}$. But I have not been able to prove the result by merely assuming convergence at every point of (a, b).

ON STURM-LIOUVILLE SERIES.

BY

K. CHANDRASEKHARAN.

[Received 4 July 1944.]

1. Consider the solutions of the Sturm-Liouville differential equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)V=0 \tag{1}$$

for an interval (a, b) of the variable x, which satisfy the the boundary-conditions $\frac{dV}{dx}-hV=0$ for $x=a$, $\frac{dV}{dx}+HV=0$ for $x=b$, where h, H are positive constants. It is assumed that g, k, l are functions of x which are positive and non-vanishing in (a, b); r is a parameter; $g, k,$ have everywhere finite derivatives; l, $(gk)^{-\frac{1}{4}}$ are of bounded variation in (a, b).

Equation (1) can then be transformed into

$$\frac{d^2U}{dz^2}+(\rho^2-l_1)U=0 \tag{2}$$

by the substitutions

$$z=\int_a^x (g/k)^{\frac{1}{2}}dx,\ \theta=(gk)^{-\frac{1}{4}},\ V=\theta U,\ r=\rho^2$$

and $$l_1=\frac{1}{\theta(gk)^{\frac{1}{2}}}\left\{l\left(\frac{k}{g}\right)^{\frac{1}{2}}\theta-\frac{d(gk)^{\frac{1}{2}}}{dz}\frac{d\theta}{dz}-(gk)^{\frac{1}{2}}\frac{d^2\theta}{dz^2}\right\},$$

with the corresponding boundary-conditions

$$\frac{dU}{dz}-h'U=0 \text{ for } z=0,\ \frac{dU}{dz}+H'U=0 \text{ for } z=\pi.$$

If $\{\phi_n(z)\}$ are a set of normalized, orthogonal functions, we can associate with any given function $f(z)$ a series $\Sigma a_r\phi_r(z)$, $a_r=\int_0^\pi f\phi_r(t)dt$, called the *Fourier ϕ-series.* In particular, if $\{\phi_n\}$ are the eigenfunctions corresponding to the eigenvalues ρ_n of equation (2), the series $\Sigma a_r\phi_r(z)$ is called the *Sturm-Liouville series* of $f(z)$.

Hobson[2] has proved that the Sturm-Liouville series of $f(z)$, where $f(z)$ is Lebesgue-integrable in $(0, \pi)$ converges to the value $\frac{1}{2}\{f(z+0)+f(z-0)\}$ at any interior point z of the interval $(0, \pi)$, if a neighbourhood of the point z exists in which the function $f(z)$ is of limited total fluctuation. In any interval in which $f(z)$ is continuous, and which is contained in the interior of an interval in which the function has limited total fluctuation, the convergence of the series to the value $f(z)$ is uniform. At the points $z=0$, $z=\pi$, the series converges to the value $f(0+0), f(\pi-0)$, if the function is of limited total fluctuation in the neighbourhoods of these points.

More generally, Haar[1] has proved that the Sturm-Liouville development of an integrable function converges or diverges at a point according as the cosine-series of this function converges or diverges at that point.

The object of this paper is to study* the behaviour of the differentiated Sturm-Liouville series, and the *conjugate* Sturm-Liouville series, on the analogy of Fourier trigonometric series.

2. Lemma 1.

$$\phi_n(z)^{(2)} = \sqrt{\frac{2}{\pi}} \cos nz \left\{1+\frac{\alpha(z, n)}{n^2}\right\} + \sin nz \left\{\frac{\alpha(z)}{n}+\frac{\alpha(z, n)}{n^2}\right\}$$

$$\phi_n'(z) = -\sqrt{\frac{2}{\pi}}\left\{n+\frac{\alpha(z, n)}{n}\right\}\sin nz + \sqrt{\frac{2}{\pi}}\left\{h'-cz+\frac{\alpha(z, n)}{n}\right\}\cos nz$$

where $|\alpha(z, n)|$, $|\alpha(z)|$ *are less than fixed numbers independent of* z *and* n.

Lemma 2. *If* $f(x)$ *has a finite jump*[5] $D(x)=f(x+0)-f(x-0)$, *the sequence*

$$\{n(b_n \cos nx - a_n \sin nx)\}$$

(*where* a_n, b_n *are the usual Fourier coefficients*) *is summable* (C, r) *to the value* D/π; *if* $f(x)$ *is of bounded variation, the same is true for* $r>0$.

* I am indebted to Dr. S. Minakshisundaram for suggesting this problem.

THEOREM I. *If $f(x)$ has a finite jump $D(x)$, and B_r^α denotes the rth Cesaro mean of order α of the series $\Sigma a_r\ \phi'_r(x)$, then* $\dfrac{B_r^\alpha}{r} \to \dfrac{D(x)}{\pi(\alpha+1)}$ *provided $\alpha > 0$; if $f(x)$ is of bounded variation, the same is true for $\alpha > -1$.*

PROOF.

$$\text{Let } b_r = \phi'_r(x)\int_0^\pi f\phi_r(t)dt,\ c_r = -\frac{2}{\pi} r \sin rx \int_0^\pi f \cos rt\, dt$$

$$\phi'_r(x)\phi_r(t) + \frac{2}{\pi} r \sin rx \cos rt$$

$$= \left\{\sqrt{\frac{2}{\pi}} \cos rt \left(1 + \frac{\alpha(t,r)}{r^2}\right) + \sin rt \left(\frac{\alpha(t)}{r} + \frac{\alpha(t,r)}{r^2}\right)\right\}$$

$$\times \left\{-\sqrt{\frac{2}{\pi}}\left(r + \frac{\alpha(x,r)}{r}\right)\sin rx + \sqrt{\frac{2}{\pi}}\left(h' - cx + \frac{\alpha(x,r)}{r}\right)\cos rx\right\}$$

$$+ \frac{2}{\pi} r \sin rx \cos rt, \text{ by Lemma 1,}$$

$$= \beta(x) \cos rt \cos rx + \gamma(t) \sin rt \sin rx + O(1/r).$$

Therefore,

$$b_r - c_r = \beta(x) \cos rx \int_0^\pi f \cos rt\, dt + \sin rx \int_0^\pi \gamma f(t) \sin rt\, dt$$

$$+ O(1/r) = o(1),$$

since the Fourier coefficients of integrable functions tend to zero. Hence, $b_r^\alpha - c_r^\alpha = o(1)$, for $\alpha > 0$ where b_r^α, c_r^α denote the Cesaro means of order α of the respective sequences. By Lemma 2, $c_r^\alpha \to D(x)/\pi$, for $\alpha > 1$ if f is integrable, and for $\alpha > 0$ if f is of bounded variation, so that $b_r^\alpha \to D(x)/\pi$, for $\alpha > 1$ if f is integrable, and for $\alpha > 0$ if f is of bounded variation.

If B_r^α denotes the Cesaro mean of order α of the series Σb_r, then $B_r^\alpha/r = b_r^{\alpha+1}/(\alpha+1)$ so that $B_r^\alpha/r \to D(x)/\pi(\alpha+1)$, for $\alpha > 0$ if f is integrable, and for $\alpha > -1$ if f is of bounded variation.

3. Let us pass back to the functions which satisfy (1). We write

$$\phi_n(z) = (gk)^{\frac{1}{4}} V_n(x);$$

then, since $dz = (g/k)^{\frac{1}{2}}dx$, we have $\int_a^b g V_m(x) V_n(x)\, dx = 0$ if $n \neq m$, $\int_a^b g \{ V_n(x) \}^2 dx = 1$. Writing $\chi(x)$ for $f(z)$, $\Sigma a_r \phi_r(z)$ becomes

$$\sum (gk)^{\frac{1}{4}} V_n(x) \int_a^b g' V_n(x') \frac{\chi(x')}{(g'k')^{\frac{1}{4}}} dx';$$

if we write $F(x)$ for $\chi(x')\,(g'k')^{-\frac{1}{4}}$ we obtain the series

$$\sum (gk)^{\frac{1}{4}} V_n(x) \int_a^b g' V_n(x')\, F(x')\, dx'.$$

LEMMA 3. *If*(5) *$f(x)$ is in L^p, $p > 1$, then, the conjugate Fourier series of $f(x)$ will itself be a Fourier series.*

LEMMA 4. *If $f(x)$ is in $L^p, p > 1$, the series*

$$\sum \frac{\phi_r'(z)}{\rho_r} \int_0^\pi f(t)\, \phi_r(t) dt \quad \textit{and} \quad -\frac{2}{\pi} \sum \sin rz \int_0^\pi f(t) \cos rt\, dt,$$

are uniformly equiconvergent in $(0, \pi)$ in the wider sense.

PROOF. $\chi_r(z, t) = \dfrac{\phi'_r(z) \phi_r(t)}{\rho_r} + \dfrac{2}{\pi} \sin rz \cos rt$

$$= \left[\sqrt{\frac{2}{\pi}} \frac{\cos rt}{r} \left\{ 1 + \frac{\alpha(t, r)}{r^2} \right\} + \frac{\sin rt}{r} \left\{ \frac{\alpha(t)}{r} + \frac{\alpha(t, r)}{r^2} \right\} \right]$$

$$\times \left[-\sqrt{\frac{2}{\pi}} \left(r + \frac{\alpha(z, r)}{r} \right) \sin rz + \sqrt{\frac{2}{\pi}} \left(h' - cz + \frac{\alpha(z, r)}{r} \right) \cos rz \right]$$

$$+ \frac{2}{\pi} \sin rz \cos rt + O(1/r^2)$$

$$= \frac{2}{\pi} (h' - cz) \frac{\cos rz \cos rt}{r} - \sqrt{\frac{2}{\pi}}\, \alpha(t) \frac{\sin rz \sin rt}{r} + O\left(\frac{1}{r^2} \right)$$

$$\int_0^\pi f(t) \sum_1^n \chi_r(z, t)\, dt = \frac{2}{\pi} (h' - cz) \sum_1^n \frac{\cos rz}{r} \int_0^\pi f(t) \cos rt\, dt$$

$$- \sqrt{\frac{2}{\pi}} \sum_1^n \frac{\sin rz}{r} \int_0^\pi \alpha(t)\, f(t) \sin rt\, dt + \sum_1^n O(1/r^2).$$

The first two series on the right are the integrated, conjugate cosine-series and sine-series of a function $f(x) \in L^p$, $p > 1$.

Since, by Lemma 3, the conjugate series are themselves Fourier series, the integrated series converge. The third series on the right is obviously convergent, and hence the lemma.

LEMMA 5. *The series*(3)

$$\sum_{1}^{\infty}\frac{a_r\phi_r(z)}{\rho_r} \text{ and } \frac{1}{\pi}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{\cos rz}{r}\int_0^{\pi}f(t)\cos rt\,dt$$

are uniformly equi-convergent in $(0, \pi)$.

LEMMA 6. *If* $F(x)\varepsilon L^p$, $p > 1$, *then the series* $\sum\frac{V_r'(x)}{\rho_r}\int_a^b gV_rF\,dt$ *converges or diverges with the conjugate cosine series of* $f(z)$.

PROOF. $$a_r=\int_0^{\pi}f(t)\phi_r(t)dt=\int_a^b gV_r(t)F(t)dt.$$

Now, $$V_r(x)=\phi_r(z)(gk)^{-\frac{1}{4}}=\phi_r(z)\,\theta(x), \text{ say, so that}$$
$$V'_r(x)=(g/k)^{\frac{1}{2}}\theta(x)\,\phi_r'(z)+\theta'(x)\phi_r(z)$$
and
$$\Sigma a_r\,V_r'(x)/\rho_r=(g/k)^{\frac{1}{2}}\theta(x)\,\Sigma a_r\phi'_r(z)/\rho_r+\theta'(x)\,\Sigma a_r\phi_r(z)/\rho_r.$$
Now if $F(x)\varepsilon L^p$, then $f(z)\varepsilon L^p$ for $p>1$, and conversely; hence, by Lemma 4, $\Sigma\, a_r\phi'_r(z)/\rho_r$ is uniformly equiconvergent, in the wider sense, with the conjugate cosine series; whereas, $\Sigma a_r\phi_r(z)/\rho_r$ converges by Lemma 5.

Hence $\Sigma a_rV_r'(x)/\rho_r$ converges or diverges along with the conjugate cosine series of $f(z)$.

DEFINITION: *If* $\Sigma a_rV_r(z)$ *is the Sturm-Liouville series of* $f(z)$, *the conjugate Sturm-Liouville series is defined to be the series* $\Sigma a_r\,V_r'(z)/\rho_r$, *where* $\{V_r'(z)/\rho_r\}$ *form a normal orthogonal set.*

Now, $\{V_r'(z)/\rho_r\}$ will form a normal orthogonal set under the following boundary-conditions,

(i) $V'(a)=0$, $V'(b)=0$
(ii) $V(a)=0$, $V(b)=0$

which are particular cases of (1).

THEOREM 2. *The conjugate Sturm-Liouville series of the function $F(x)$ belonging to the class $L^p, p > 1$ converges or diverges along with the conjugate cosine series of the function $f(z)$.*

This is an immediate consequence of Lemma 6.

THEOREM 3. *If $F(x) \epsilon L^p, p > 1$, and $\Sigma a_r V_r'(x)/\rho_r$ is the conjugate Sturm-Liouville series, then there exists a function $G(x) \epsilon L^p$ whose Fourier ψ-series is $\Sigma a_r V_r'(x)/\rho_r$, $\psi_r = V_r'/\rho_r$.*

PROOF. By Lemma 6, if $S_n = \Sigma a_r V_r'(x)/\rho_r$ and $\sigma_n = -\frac{2}{\pi}\sum \text{Sin } rz \int_0^\pi f \cos rt\, dt$, then $S_n(x) - \sigma_n(z) \rightarrow \phi(x)$ uniformly, where $\phi(x)$ is a continuous function.

σ_n is the partial sum of the conjugate cosine series of $f(z) \epsilon L^p, p > 1$; and hence σ_n is the partial sum of a Fourier series, so that if $\bar{f}$ is *its* sum[5]

$$\int_0^\pi |\sigma_n - \bar{f}|^p dt \rightarrow 0.$$

Now, $\quad S_n = \sigma_n + \phi(x) + R(n)$, where $R(n) = o(1)$,
so that

$$\left\{\int_0^\pi |S_n - \bar{f} - \phi(t)|^p dt\right\}^{1/p} \leqslant \left\{\int_0^\pi |\sigma_n - \bar{f}|^p dt\right\}^{1/p} + \left\{\int_0^\pi |R_n|^p dt\right\}^{1/p}, \; p > 1,$$

which leads to $\quad \int_0^\pi |S_n - G|^p dt \rightarrow 0, \quad$ where $G = \bar{f} - \phi$.

Hence, $\Sigma a_r V_r'/\rho_r$ is the Fourier ψ-series of $G(x) \; \epsilon L^p, p > 1$.

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SOME FURTHER ALGEBRAIC IDENTITIES.

BY

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[Received in revised form 18 November 1944].

1. Notations: If x and c be any numbers an $|x-c| \equiv (\text{mod } b)$, then

$$\begin{bmatrix} x \\ c \end{bmatrix}_b = (a^x-1)(a^{x-b}-1)(a^{x-2b}-1)\ldots(a^c-1), \text{ if } x>c \text{ and}$$

$= 1$, if $x<c$. If x and c are positive integers,

we write $\begin{bmatrix} x \\ c \end{bmatrix}$ for $\begin{bmatrix} x \\ c \end{bmatrix}_1$. If n is a positive integer, we write

$${}_xS_n = \frac{\begin{bmatrix} x \\ x-n+1 \end{bmatrix}_1 a^{\frac{1}{2}n(n-1)}}{\begin{bmatrix} n \\ 1 \end{bmatrix}}$$

If x and n are both positive integers, we write

$${}_xS_n = \frac{\begin{bmatrix} x \\ 1 \end{bmatrix} a^{\frac{1}{2}n(n-1)}}{\begin{bmatrix} n \\ 1 \end{bmatrix}\begin{bmatrix} x-n \\ 1 \end{bmatrix}}.$$

It has been shown elsewhere* that ${}_xS_n$ is the sum of the products of the factors $1, a, a^2, \ldots, a^{x-1}$ taken n at a time and hence ${}_xS_n = a^{\frac{1}{2}n(n-1)}$.

We have also proved that

$$\sum_{m=0}^{n} (-)^m \begin{bmatrix} 2n-1-2m \\ 1 \end{bmatrix}_2 {}_{2n}S_{2m} = (-)^n \tag{1}$$

$$\sum_{m=0}^{n} (-)^m \begin{bmatrix} 2n-1-2m \\ 1 \end{bmatrix}_2 {}_{2n+1}S_{2m+1} = (-)^n. \tag{2}$$

If $1, a, a^2, a^3, \ldots$ be the successive multipliers then

$$\sum_{m=0}^{n} (-)^m u_m \, {}_nS_m \tag{3}$$

* *Bull. Calcutta Math. Soc.* 14 (1923-4), 91-106.

is the first element of the n th order of differences obtained from the series $u_0, u_1, u_2 \ldots$

2. THEOREM I. *Let*

$$A_5 = \begin{vmatrix} {}_6S_1 & {}_6S_2 & {}_6S_3 & {}_6S_4 & {}_6S_6 - 1 \\ 1 & {}_5S_1 & {}_5S_2 & {}_5S_3 & {}_5S_5 - 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & {}_2S_2 - 1 \end{vmatrix}.$$

Then A_n, *formed similarly*, $= 0$ *or* $\begin{bmatrix} n \\ 1 \end{bmatrix}_2$, *according as n is even or odd.*

The proof follows by induction, since

$$A_5 = {}_6S_1A_4 - {}_6S_2A_3 + {}_6S_3A_2 - {}_6S_4A_1 + {}_6S_6 - 1$$

$$= -\sum_{m=0}^{3} (-)^m \begin{bmatrix} 5-2m \\ 1 \end{bmatrix}_2 {}_6S_{2m} - 1 + \begin{bmatrix} 5 \\ 1 \end{bmatrix}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}_2, \text{ by (1)}.$$

3. We now prove a number of theorems by using (3).

THEOREM 2.

$$\sum_{m=1}^{n} (-)^{m-1} {}_nS_m/(a^m - 1) = \frac{1}{a-1} + \frac{a^2}{a^2-1} + \frac{a^3}{a^3-1} + \ldots n \text{ terms.}$$

This follows by taking $u_0 = 0$ and $u_m = 1/(a^m - 1)$ in (3).

THEOREM 3.

$$\sum_{m=0}^{r-k} (-)^m \begin{bmatrix} m \\ 1 \end{bmatrix} a^{km} {}_{r-k}S_m \Big/ \begin{bmatrix} k+m \\ k+1 \end{bmatrix}_1 = (a^k - 1)/(a^r - 1) \qquad (4)$$

$r-k$ being a positive integer.

PROOF. Take $u_m = \begin{bmatrix} m \\ 1 \end{bmatrix} a^{km} \Big/ \begin{bmatrix} k+m \\ k+1 \end{bmatrix}_1$ in (3).

Multiplying (4) by $\dfrac{\begin{bmatrix} r \\ 1 \end{bmatrix} {}_kS_k}{\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} r-k \\ 1 \end{bmatrix}}$ and using our definitions, we have:

COR. 1. $$\sum_{m=0}^{r-k} (-)^m {}_rS_{k+m} = {}_{r-1}S_{k-1}a^{k-1} \qquad (5)$$

where r and k are both positive integers and $r > k$.

$$\text{Cor. 2.}\quad \sum_{m=0}^{k-1} (-)^m {}_rS_m = (-)^{k-1} {}_{r-1}S_{k-1} a^{k-1} \tag{6}$$

r being a positive integer $> k$.

This follows from (5), since $\sum\limits_{m=0} (-)^m {}_rS_m = 0$.

Cor. 3. *Let*

$$r_4 = \begin{vmatrix} {}_rS_1 & {}_rS_2 & {}_rS_3 & {}_{r-1}S_3a^3 \\ 1 & {}_{r-1}S_1 & {}_{r-1}S_2 & {}_{r-2}S_2a^2 \\ & 1 & {}_{r-2}S_1 & {}_{r-3}S_1a \\ & & 1 & 1 \end{vmatrix}$$

where r, on either side, is the same positive integer. Then r_n, similarly defined, $= 1$, $1 < n \leqslant r$. (7)

The proof follows from (6). It follows from (7) that $r_2 = r_3 = r_4 = \ldots = r_r = 1$.

Theorem 4.

$$\sum_{m=0}^{r} \frac{{}_{m+1}S_{m+1}a^{m(x-r)}}{\begin{bmatrix} x-r+m \\ x-r+1 \end{bmatrix}_1 \begin{bmatrix} y-m \\ y-r+1 \end{bmatrix}_1} {}_rS_m = \frac{\begin{bmatrix} x+y \\ x+y-r+1 \end{bmatrix}_1}{\begin{bmatrix} x \\ x-r+1 \end{bmatrix}_1 \begin{bmatrix} y \\ y-r+1 \end{bmatrix}_1}, \tag{8}$$

where x and y are any numbers.

This again follows from (3), taking

$$u_m = (-)^m \frac{{}_{m+1}S_{m+1}a^{m(x-r)}}{\begin{bmatrix} x-r+m \\ x-r+1 \end{bmatrix}_1 \begin{bmatrix} y-m \\ y-r+1 \end{bmatrix}_1}.$$

$$\text{Cor. (i)}\qquad \sum_{m=0}^{r} {}_xS_{r-m}\, {}_yS_m a^{xm} = {}_{x+y}S_r. \tag{9}$$

For, by the definition of ${}_xS_n$, the left side

$$= \sum_{m=0}^{r} \frac{\begin{bmatrix} x \\ x-r+m+1 \end{bmatrix}_1 \begin{bmatrix} y \\ y-m+1 \end{bmatrix}_1 {}_{r-m}S_{r-m} \cdot {}_mS_m a^{xm}}{\begin{bmatrix} r-m \\ 1 \end{bmatrix} \begin{bmatrix} m \\ 1 \end{bmatrix}}.$$

$$= \frac{\begin{bmatrix} x \\ x-r+1 \end{bmatrix}_1 \begin{bmatrix} y \\ y-r+1 \end{bmatrix}_1 {}_rS_r}{\begin{bmatrix} r \\ 1 \end{bmatrix}} \times \text{the left side of (8).}$$

When x and y are positive integers, (9) reduces to Gupta's result,* and when $a = 1$, this becomes the familiar result

$$_{m+n}C_r = \sum_{p=0}^{r} {}_mC_p \,.\, {}_nC_{r-p}.$$

COR. (ii) $$\sum_{m=0}^{n} {}_nS_{n-m} \,.\, {}_nS_m a^{n(n-m)} = {}_{2n}S_n. \qquad (10)$$

This follows from (9), writing it in the reverse order.

THEOREM 5.

$$\sum_{m=0}^{k} {}_xS_{m+1} a^m {}_kS_m = \begin{bmatrix} k+x \\ x \end{bmatrix}_1 \Big| \begin{bmatrix} k+1 \\ 1 \end{bmatrix}. \qquad (11)$$

PROOF. Take $u_m = (-)^m {}_xS_{m+1} a^m$ in (3).

COR. $$\sum_{m=0}^{n-1} {}_{m+1}S_1 ({}_nS_{m+1})^2 = {}_{2n-1}S_n \,.\, {}_nS_1 / {}_nS_n, \qquad (12)$$

where n is a positive integer.

For, by definition, the left side reduces to the left side of (11), multiplied by $(a^n-1)/(a-1)$ with $k = n-1$.

4. We now consider a generalization of the Binomial theorem.

Let $f(n, x) = 1 - {}_nS_1 x + {}_nS_2 x^2 - {}_nS_3 x^3 + \dots$

which is easily seen to converge absolutely if $|x| < a$ when $a > 1$, and if $|x| < a^{-n}$ when $a < 1$, n being any number. If n is a positive integer, $f(n, x) = (1-x)\,(1-ax) \times (1-a^2x)\dots(1-a^{n-1}x)$. (13)

THEOREM 6. *$f(n,x) f(m, a^n x) = f(m+n, x)$ for all values of m and n.*

The proof follows the usual lines, by multiplying the two series on the left-side.

* Hans Raj Gupta, *J.I.M.S.* (2), 5 (1941), 27-31.

Hence

$$f(n,x)\,f(m,a^n x)\,f(k,a^{n+m}x)\,f(l,a^{n+m+k}x)\ldots r \text{ factors}$$
$$=f(n+m+k+\ldots,\ x).$$

In particular

$$f\left(\frac{p}{r},x\right)f\left(\frac{p}{r},a^{\frac{p}{r}}x\right)f\left(\frac{p}{r},\ a^{\frac{2p}{r}}x\right)\ldots r \text{ factors}=f(p,\,x) \quad (14)$$

and
$$f(n,\ x)\,f(-n,\ a^n x)=1. \quad (15)$$

It is easily seen that all the series on the left side of (14) converge absolutely when $|x|<a^{-p/r}$ if $a<1$ and when $|x|<a^{1-(r-2)p/r}$ if $a>1$. When $a=1$, the results (13), (14) and (15) reduce to the Binomial theorem.

5. We add some further results involving the expression ${}_xS_n$.

THEOREM 7. If $k=$ integral part of $\frac{1}{2}n$,

$$\sum_{m=0}^{k}\frac{{}_nS_{2m}a^{2m}}{a^{2m+1}-1}=\frac{1}{a^{n+1}-1}\prod_{r=1}^{n}\left(1+a^r\right). \quad (16)$$

PROOF. Since $\sum\limits_{m=0}^{n+1}(-)^m{}_{n+1}S_m=0$, we have

$$1+{}_{n+1}S_2+{}_{n+1}S_4+\ldots={}_{n+1}S_1+{}_{n+1}S_3+\ldots$$
$$=\tfrac{1}{2}\sum_{m=0}^{n+1}{}_{n+1}S_m=\prod_{r=1}^{n}\left(1+a^r\right).$$

But ${}_{n+1}S_{2m+1}=\dfrac{a^{n+1}-1}{a^{2m+1}-1}\,{}_nS_{2m}a^{2m}$. The theorem follows by adding these equations as m varies.

THEOREM 8.

$$\sum_{m=0}^{p}(-)^m\frac{\begin{bmatrix}3p-m-1\\2p-m\end{bmatrix}}{a^{k+m}-1}\,{}_pS_m=\frac{\begin{bmatrix}3p+k-1\\2p+k\end{bmatrix}_1\begin{bmatrix}p\\1\end{bmatrix}}{\begin{bmatrix}k+p\\k\end{bmatrix}_1}.$$

PROOF. Let $p=3$, for example. Writing λ for a^k and multiplying the whole equation by $(a^3\lambda-1)(a^2\lambda-1)\times(a\lambda-1)(\lambda-1)$, we have an equation of the third degree in λ. It can be shown that this is satisfied by $\lambda=1,\ 1/a,\ 1/a^2$ or $1/a^3$. Hence it is an identity.

ON SOME TRANSFORMATIONS OF THE GENERALIZED HYPERGEOMETRIC SERIES

BY

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[Received in revised form 12 September, 1944.]

1. The following results are well known(1).

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right] $$

$$ =\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-d-e)} \text{(Dougall)}. \tag{1} $$

$$ {}_3F_2\left[\begin{matrix} a, & b, & c; \\ & 1+a-b, & 1+a-c \end{matrix}\right] $$

$$ =\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a+c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)} \text{(Dixon)}. \tag{2} $$

$$ {}_2F_1[\alpha, \beta; \gamma;] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \text{ (Gauss)}. \tag{3} $$

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ \frac{1}{2}(a+b+1), 2c \end{matrix}\right] $$

$$ =\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)} \text{(Watson)}. \tag{4} $$

Multiplying (1) by

$$ \frac{\Gamma(a)\Gamma(1+\frac{1}{2}a)\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(\frac{1}{2}a)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)} $$

we get after slight rearrangement

$$ \frac{\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-d-e)} $$

$$ =\frac{\Gamma(\frac{1}{2}a)}{\Gamma(a)\Gamma(1+\frac{1}{2}a)\Gamma(c)\Gamma(d)\Gamma(e)} $$

$$ \times \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(1+\frac{1}{2}a+r)\Gamma(c+r)\Gamma(d+r)\Gamma(e+r)}{\Gamma(r+1)\Gamma(\frac{1}{2}a+r)\Gamma(1+a-c+r)\Gamma(1+a-d+r)\Gamma(1+a-e+r)}. \tag{5} $$

Write $1+a-c-d=\alpha_1$, $1+a-c=\rho_1$, $1+a-d=\rho_2$ and $e=-s$. Multiplying (5) by $\frac{1}{2\pi i}\frac{\Gamma(-s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)\,(-x)^s}{\Gamma(\rho_3+s)\ldots\Gamma(\rho_q+s)}$, we get

$$\frac{1}{2\pi i}\frac{\Gamma(-s)\Gamma(\alpha_1+s)\ldots\Gamma(\alpha_p+s)\,(-x)^s}{\Gamma(\rho_1+s)\Gamma(\rho_2+s)\ldots\Gamma(\rho_q+s)}$$

$$=\frac{1}{2\pi i}\frac{\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)]\Gamma(\rho_1+\rho_2-\alpha_1)\Gamma(\alpha_1)}{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)]\Gamma(\rho_1+\rho_2-\alpha_1-1)\Gamma(\rho_1-\alpha_1)\Gamma(\rho_2-\alpha_1)}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)+r]\Gamma(\rho_1+\rho_2-\alpha_1-1+r)\Gamma(\rho_{_}-\alpha_1+r)}{\Gamma(r+1)\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)+r]\Gamma(\rho_1+r)\Gamma(\rho_2+r)}$$

$$\times\frac{\Gamma(\rho_1-\alpha_1+r)\Gamma(r-s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)(-x)^s}{\Gamma(\rho_1+\rho_2-\alpha_1+s+r)\Gamma(\rho_3+s)\ldots\Gamma(\rho_q+s)}. \quad (6)$$

Let us integrate both sides of (6) along a contour C_N which encloses $(N+1)$ poles only of $\Gamma(-s)$ but none of the other poles of the integrands, so that $R(\alpha_i)>0$. The contour C_N is taken to be a triangle (obtained by joining the point $N+k$ on the x-axis, where $0<k<1$, to two points on the y-axis equidistant from the origin) except for a semicircular loop round the origin so as to exclude the origin. In the limit when $N\to\infty$ (through positive integral values) C_N becomes C, a contour which consists of two infinite lines starting parallel to the positive x-axis from two points on the y-axis (equidistant from the origin) and the finite portion of the y-axis between them.

Since $\Gamma(\alpha+s)\sim\exp\{(s+\alpha-\frac{1}{2})\log s-s+\frac{1}{2}\log 2\pi\}+O(1/s)$, when $|\arg s|<\pi$ and $I(s)$ is finite as $s\to\infty$ along C, we have $\frac{\Gamma(r-s)\Gamma(\alpha_2+s)\ldots\ldots\Gamma(\alpha_p+s)(-x)^s}{\Gamma(\rho_1+\rho_2-\alpha_1+r+s)\Gamma(\rho_3+s)\ldots\ldots\Gamma(\rho_q+s)}\sim\exp[(p-q-1)\times s\log s-(p-q-1)s+s\log|x|+O(\log s)]$ and hence the left side $\to 0$ exponentially, as $R(s)\to\infty$ along C, for all x when $p-q-1<0$, and for $|x|<1$ when $p-q-1=0$. The right side of (6) converges uniformly with respect to s if $R(s+\alpha_1)>0$. Hence, on making $N\to\infty$ and then interchanging the order of integration and summation in (6) we get

$$
{}_pf_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\ x\\ \rho_1,\ldots,\rho_q\end{matrix}\right)
$$

$$
=\frac{\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)]\,\Gamma(\rho_1+\rho_2-\alpha_1)\Gamma(\alpha_1)}{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)]\Gamma(\rho_1+\rho_2-\alpha_1-1)\Gamma(\rho_1-\alpha_1)\Gamma(\rho_2-\alpha_1)}
$$

$$
\times\sum_{r=0}^{\infty}\frac{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)+r]\Gamma(\rho_1+\rho_2-\alpha_1-1+r)}{\Gamma(r+1)\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)+r]}
$$

$$
\times\frac{\Gamma(\rho_1-\alpha_1+r)\Gamma(\rho_2-\alpha_1+r)}{\Gamma(\rho_1+r)\Gamma(\rho_2+r)}
$$

$$
\times\frac{1}{2\pi i}\int_C\frac{\Gamma(r-s)\Gamma(\alpha_2+s)\ldots\ldots\Gamma(\alpha_p+s)(-x)^s ds}{\Gamma(r+1)\Gamma(\rho_1+\rho_2-\alpha_1+r+s)\Gamma(\rho_3+s)\ldots\Gamma(\rho_q+s)},
$$

where
$$
{}_pf_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\ x\\ \rho_1,\ldots,\rho_q\end{matrix}\right)
$$

$$
=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\ x\\ \rho_1,\ldots,\rho_q\end{matrix}\right],\ \text{provided}\ R(s+\alpha_1)>0,
$$

and either $p-q-1<0$ for all x, or $p-q-1=0$ for $|x|<1$.

Let us consider the integral on the right side. Regarding C to be the limit of C_N mentioned above, we observe that

$$
\frac{1}{2\pi i}\int_C\frac{\Gamma(r-s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)(-x)^s ds}{\Gamma(r+1)\Gamma(\rho_1+\rho_2-\alpha_1+r+s)\Gamma(\rho_3+s)\ldots\Gamma(\rho_q+s)}
$$

$$
=\operatorname*{Lt}_{N\to\infty}\frac{1}{2\pi i}\int_{C_N}\frac{\Gamma(r-s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)(-x)^s ds}{\Gamma(r+1)\Gamma(\rho_1+\rho_2-\alpha_1+r+s)\Gamma(\rho_3+s)\ldots\Gamma(\rho_q+s)}
$$

$$
=(-x)^r\,{}_{p-1}f_{q-1}\left(\begin{matrix}\alpha_2+r,\alpha_3+r,\ldots\alpha_p+r;\ x\\ \rho_1+\rho_2-\alpha_1+2r,\rho_3+r,\ldots\rho_q+r\end{matrix}\right)
$$

Hence for all x when $p-q-1<0$ and for $|x|<1$ when $p-q-1=0$, we have

$$
{}_pf_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\ x\\ \rho_1\ldots,\rho_q\end{matrix}\right)
$$

$$
=\frac{\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)]\Gamma(\rho_1+\rho_2-\alpha_1)\Gamma(\alpha_1)}{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)]\Gamma(\rho_1+\rho_2-\alpha_1-1)\Gamma(\rho_1-\alpha_1)\Gamma(\rho_2-\alpha_1)}
$$

$$
\times\sum_{r=0}^{\infty}\frac{\Gamma[\frac{1}{2}(1+\rho_1+\rho_2-\alpha_1)+r]\Gamma(\rho_1+\rho_2-\alpha_1-1+r)}{\Gamma(r+1)\Gamma[\frac{1}{2}(\rho_1+\rho_2-\alpha_1-1)+r]}
$$

$$\times \frac{\Gamma(\rho_1-\alpha_1+r)\Gamma(\rho_2-\alpha_1+r)(-x)^r}{\Gamma(\rho_1+r)\Gamma(\rho_2+r)}$$

$$\times\ {}_{p-1}f_{q-1}\left(\begin{matrix}\alpha_2+r,\ \alpha_3+r,\ldots\ldots\ \alpha_p+r;\ x\\ \rho_1+\rho_2-\alpha_1+2r,\ \rho_3+r,\ldots\ldots\ \rho_q+r\end{matrix}\right),\ R(s+\alpha_1)>0 \quad (7)$$

That the right side of (7) converges may be seen as follows. If u_n is the nth term of

$${}_{p-1}F_{q-1}\left[\begin{matrix}\alpha_2+r,\ \alpha_3+r,\ldots\ldots,\ \alpha_p+r;\ x\\ \rho_1+\rho_2-\alpha_1+2r,\ \rho_3+r,\ldots\ldots,\ \rho_q+r\end{matrix}\right],$$

then according as $p-q-1=0$, or <0,

$$|u_{n+1}/u_n| \to |x| \text{ or } 0 \text{ as } n\to\infty,$$

and noting that $|u_{n+1}|<\rho\,|u_n|$, $0<\rho<1$, for $n\leqslant m$, (say) we get

$$|{}_{p-1}F_{q-1}|<k\,|u_\nu| = k|c_\nu|\ |x|^\nu,$$

where c_n is the co-efficient of x^n in ${}_{p-1}F_{q-1}$ and $|c_\nu|$ is the greatest of the terms 1, $|c_1|,\ldots|c_{m-1}|$, $|c_m|/(1-\rho)$ and k is independent of r.

It follows that the absolute value of the general term in (7) is less than the absolute value of the general term of a Hypergeometric series of the form ${}_{p+3}f_{q+3}$ with argument $x/4$, which evidently converges absolutely for all x when $p-q-1<0$ and for $|x|<1$ when $p-q-1=0$.

The restrictions on the α's may be removed by analytic continuation. Now (7) also holds for $|x|=1$ when $p-q-1=0$. This we prove by using Abel's theorem on continuity of power series by first proving that both sides of (7) converge when $|x|=1$.

Taking $x=1$, we observe that the series on the left side of (7) converges if $R(\Sigma\rho_i-\Sigma\alpha_i)>0$, or if

$$R\Big[\rho_1+\rho_2-\alpha_1+2r+\sum_{i=3}^{q}(\rho_i+r)-\sum_{i=2}^{q+1}(\alpha_i+r)\Big]>0.$$

Hence it follows[(2)] that $\lim n\left\{|u_{n+1}/u_n|-1\right\}=-1-c$, where c(independent of r) is positive and u_n denotes the nth term of the series

$$ {}_{q+1}F_q\left[\begin{matrix}\alpha_2+r, & \ldots\ldots\ldots, & \alpha_{q+1}+r; \\ \rho_1+\rho_2-\alpha_1+2r, & \rho_3+r, \ldots, & \rho_q+r\end{matrix}\right]. $$

From this we can show(2) that $|u_n| < An^{-1-\frac{1}{2}c}$, for all n, where A is a constant chosen suitably. Hence

$$ {}_{p-1}f_{q-1}\left(\begin{matrix}\alpha_2+r, \ldots, \alpha_{q+1}+r; x \\ \rho_1+\rho_2-\alpha_1+2r, \rho_3+r \ldots \rho_q+r\end{matrix}\right) $$

$$ < k\left|\frac{\Gamma(\alpha_2+r) \quad \ldots\ldots \quad \Gamma(\alpha_{q+1}+r)}{\Gamma(\rho_1+\rho_2-\alpha_1+2r)\Gamma(\rho_3+r)\ldots\Gamma(\rho_q+r)}\right|, $$

k being independent of r.

Hence the absolute value of the general term of the series on the right of (7) is less than that of the general term of an absolutely convergent Hypergeometric series of the form ${}_{q+3}f_{q+2}$ with argument $\frac{1}{4}$.

Thus both sides of (7) converge for $x=1$ provided $R(\Sigma\rho_i-\Sigma\alpha_i)>0$, and similarly for $x=-1$ provided $R(\Sigma\rho_i-\Sigma\alpha_i+1)>0$.

When $p>q+1$, the left side of (7) is divergent unless either $x=0$ or one of the α's is a negative integer

We have considered $p=q+1$ in our subsequent discussion.

From (7) we deduce the following results. Put $x=1$, $p=4$, $q=3$, $2\rho_3=\alpha_2+\alpha_3+1$ and $2\alpha_4=\rho_1+\rho_2-\alpha_1$ and apply Watson's theorem (4). We get after simplification

$$ {}_4F_3\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4; \\ \rho_1, \alpha_1+2\alpha_4-\rho_1, \frac{1}{2}(\alpha_2+\alpha_3+1)\end{matrix}\right] $$

$$ =\frac{\Gamma[\frac{1}{2}(1-\alpha_2-\alpha_3)+\alpha_4]\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)].\ 2^{\alpha_2+\alpha_3-2}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} $$

$$ \times\left\{\frac{\Gamma(\alpha_4+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}{\Gamma(\frac{1}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_3+\frac{1}{2})}\right. $$

$$ \times\ {}_9F_8\left[\begin{matrix}\alpha_4-\frac{1}{2}, \frac{1}{2}\alpha_4+\frac{3}{4}, \alpha_4, \alpha_4-\frac{1}{2}\rho_1, \alpha_4-\frac{1}{2}\rho_1+\frac{1}{2}, \frac{1}{2}(\rho_1-\alpha_1), \\ \frac{1}{2}\alpha_4-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}(\rho_1+1), \frac{1}{2}\rho_1, \alpha_4+\frac{1}{2}(\alpha_1-\rho_1+1),\end{matrix}\right. $$

$$ \left.\begin{matrix}\frac{1}{2}(\rho_1-\alpha_1+1), \frac{1}{2}\alpha_2, \frac{1}{2}\alpha_3; \\ \alpha_4+\frac{1}{2}(\alpha_1-\rho_1), \alpha_4-\frac{1}{2}\alpha_2+\frac{1}{2}, \alpha_4-\frac{1}{2}\alpha_3+\frac{1}{2}\end{matrix}\right] $$

$$ -\frac{(\rho_1-\alpha_1)(2\alpha_4-\rho_1)\Gamma(\alpha_4+\frac{3}{2})\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}{\rho_1(2\alpha_4+\alpha_1-\rho_1)\Gamma(\frac{3}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_2+1)\Gamma(\alpha_4-\frac{1}{2}\alpha_3+1)} $$

$$
{}_9F_8\left[\begin{matrix}\alpha_4+\frac{1}{2}, \frac{1}{2}\alpha_4+\frac{5}{4}, \alpha_4, \alpha_4-\frac{1}{2}\rho_1+1, \alpha_4-\frac{1}{2}\rho_1+\frac{1}{2}, \frac{1}{2}(\rho_1-\alpha_1+1), \\ \frac{1}{2}\alpha_4+\frac{1}{4}, \frac{3}{2}, \frac{1}{2}(\rho_1+1), \frac{1}{2}\rho_1+1, \alpha_4+\frac{1}{2}(\alpha_1-\rho_1)+1,\end{matrix}\right.
$$

$$
\left.\left.\begin{matrix}\frac{1}{2}(\rho_1-\alpha_1)+1, \frac{1}{2}\alpha_2+\frac{1}{2}, \frac{1}{2}\alpha_3+\frac{1}{2}: \\ \alpha_4+\frac{1}{2}(\alpha_1-\rho_1+1), \alpha_4-\frac{1}{2}\alpha_2+1, \alpha_4-\frac{1}{2}\alpha_3+1\end{matrix}\right]\right\}, \tag{8}
$$

the two series ${}_9F_8$ being "well poised", terminating when $\rho_1-\alpha_1$ is a negative integer. If $\rho_1=2\alpha_4$, we get back Watson's theorem.

In (8) let us write $\rho_1=\alpha_1/(1-x)$ and first make $\alpha_4\to\infty$ and $\alpha_1\to\infty$ independently of each other. Taking the limits term by term in each limiting process which can be justified[3], we have

$$
\begin{aligned}
&{}_2F_1[\alpha_2, \alpha_3; \tfrac{1}{2}(\alpha_2+\alpha_3+1); \tfrac{1}{2}(1-x)] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]}{\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}\, {}_2F_1[\tfrac{1}{2}\alpha_2, \tfrac{1}{2}\alpha_3; \tfrac{1}{2}; x^2] \\
&+ \frac{x\Gamma(-\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]}{\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}\, {}_2F_1[\tfrac{1}{2}\alpha_2+\tfrac{1}{2}, \tfrac{1}{2}\alpha_3+\tfrac{1}{2}; \tfrac{3}{2}; x^2].
\end{aligned} \tag{9}
$$

Again, let $\alpha_1\to\infty$ in (8) and we have

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_2, \alpha_3, \alpha_4; \\ \rho_1, \frac{1}{2}(\alpha_2+\alpha_3+1)\end{matrix}\right] \\
&= \frac{\Gamma[\frac{1}{2}(1-\alpha_2-\alpha_3)+\alpha_4]\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]2^{\alpha_2+\alpha_3-2}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \\
&\times\left\{\frac{\Gamma(\alpha_4+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}{\Gamma(\frac{1}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_3+\frac{1}{2})}\right. \\
&\times {}_7F_6\left[\begin{matrix}\alpha_4-\frac{1}{2}, \frac{1}{2}\alpha_4+\frac{3}{4}, \alpha_4, \alpha_4-\frac{1}{2}\rho_1, \alpha_4-\frac{1}{2}\rho_1+\frac{1}{2}, \frac{1}{2}\alpha_2, \frac{1}{2}\alpha_3; \\ \frac{1}{2}\alpha_4-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\rho_1+\frac{1}{2}, \frac{1}{2}\rho_1, \alpha_4-\frac{1}{2}\alpha_2+\frac{1}{2}, \alpha_4-\frac{1}{2}\alpha_3+\frac{1}{2}\end{matrix}\right] \\
&+\frac{(2\alpha_4-\rho_1)\Gamma(\alpha_4+\frac{3}{2})\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}{\rho_1\quad\Gamma(\frac{3}{2})\Gamma(\alpha_4-\frac{1}{2}\alpha_2+1)\Gamma(\alpha_4-\frac{1}{2}\alpha_3+1)} \\
&\times {}_7F_6\left[\begin{matrix}\alpha_4+\frac{1}{2}, \frac{1}{2}\alpha_4+\frac{5}{4}, \alpha_1, \alpha_4-\frac{1}{2}\rho_1+\frac{1}{2}, \\ \frac{1}{2}\alpha_4+\frac{1}{4}, \frac{3}{2}, \frac{1}{2}\rho_1+1,\end{matrix}\right. \\
&\qquad\left.\left.\begin{matrix}\alpha_4-\frac{1}{2}\rho_1+1, \frac{1}{2}\alpha_2+\frac{1}{2}, \frac{1}{2}\alpha_3+\frac{1}{2}; \\ \frac{1}{2}\rho_1+\frac{1}{2}, \alpha_4-\frac{1}{2}\alpha_2+1, \alpha_4-\frac{1}{2}\alpha_3+1\end{matrix}\right]\right\},
\end{aligned} \tag{10}
$$

a result in agreement with Mitra's result[4]. In (8) putting $\rho_1-\alpha_1=-n$ and then making $\alpha_4\to\infty$, we get the following result, previously obtained by Fox[5] in another way,

$$
{}_3F_2\left[\begin{matrix}\rho_1+n, \alpha_2, \alpha_3; \tfrac{1}{2}\\ \rho_1, \tfrac{1}{2}(\alpha_2+\alpha_3+1)\end{matrix}\right]
$$

$$
=\frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]}{\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}\,{}_4F_3\left[\begin{matrix}-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}\alpha_2, \tfrac{1}{2}\alpha_3;\\ \tfrac{1}{2}, \tfrac{1}{2}\rho_1, \tfrac{1}{2}\rho_1+\tfrac{1}{2}\end{matrix}\right]
$$

$$
-\frac{n}{\rho_1}\,\frac{\Gamma(-\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]}{\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}
$$

$$
\times\,{}_4F_3\left[\begin{matrix}-\tfrac{1}{2}n+\tfrac{1}{2}, -\tfrac{1}{2}n+1, \tfrac{1}{2}\alpha_2+\tfrac{1}{2}, \tfrac{1}{2}\alpha_3+\tfrac{1}{2};\\ \tfrac{3}{2}, \tfrac{1}{2}\rho_1+\tfrac{1}{2}, \tfrac{1}{2}\rho_1+1\end{matrix}\right], \qquad (11)
$$

${}_4F_3$ terminating when n is a positive integer.

2. Using Dixon's theorem (2) and proceeding as before we obtain

$$
{}_pf_q\left(\begin{matrix}\alpha_1, \ldots, \alpha_p; x\\ \rho_1, \ldots \rho_q\end{matrix}\right)
$$

$$
=\tfrac{1}{2}\frac{\Gamma(\alpha_1)}{\Gamma(\rho_1-\alpha_1)\Gamma(\rho_1-1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-2+r)\Gamma(\rho_1-\alpha_1+r)\,(-x)^r}{\Gamma(r+1)\Gamma(\rho_2+r)}
$$

$$
\times\,{}_{p-1}f_{q-1}\left(\begin{matrix}\alpha_2+r, \alpha_3+r, \ldots, \alpha_p+r; x\\ 2\rho_1-1+2r, \rho_3+r, \ldots, \rho_q+r\end{matrix}\right), \qquad (12)
$$

where $\rho_2=\alpha_1+\rho_1-1$ and $p<q+1$ for all x and $p=q+1$ for $|x|\leqslant 1$. Taking $x=1$, $p=4$, $q=3$, $2\rho_1=2\alpha_4+1$, $\rho_3=\frac{1}{2}(\alpha_2+\alpha_3+1)$ and applying Watson's theorem (4), we get after simplication

$$
{}_4F_3\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \rho_1-\tfrac{1}{2};\\ \rho_1, \rho_1+\alpha_1-1, \tfrac{1}{2}(\alpha_2+\alpha_3+1)\end{matrix}\right]
$$

$$
=\frac{\Gamma\{\frac{1}{2}(\alpha_2+\alpha_3+1)\}\Gamma(\rho_1)\Gamma(\rho_1-\frac{1}{2}\alpha_2-\frac{1}{2}\alpha_3).\,2^{\alpha_2+\alpha_3-2}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}
$$

$$
\times\left\{\frac{\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}{\Gamma(\frac{1}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_2)\Gamma(\rho_1-\frac{1}{2}\alpha_3)}\right.
$$

$$
\times\,{}_6F_5\left[\begin{matrix}\rho_1-1, \rho_1-\tfrac{1}{2}, \tfrac{1}{2}\alpha_2, \tfrac{1}{2}\alpha_3, \tfrac{1}{2}(\rho_1-\alpha_1), \tfrac{1}{2}(\rho_1-\alpha_1+1);\\ \tfrac{1}{2}, \rho_1-\tfrac{1}{2}\alpha_2, \rho_1-\tfrac{1}{2}\alpha_3, \tfrac{1}{2}\rho_1+\tfrac{1}{2}\alpha_1, \tfrac{1}{2}\rho_1+\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\end{matrix}\right]
$$

$$
-\frac{(\rho_1-1)(\rho_1-\alpha_1)\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}{(\rho_1+\alpha_1-1)\Gamma(\frac{3}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_3+\frac{1}{2})}
$$

$$
\left.\times\,{}_6F_5\left[\begin{matrix}\rho_1, \rho_1-\tfrac{1}{2}, \tfrac{1}{2}\alpha_2+\tfrac{1}{2}, \tfrac{1}{2}\alpha_3+\tfrac{1}{2}, \tfrac{1}{2}(\rho_1-\alpha_1+1), \tfrac{1}{2}(\rho_1-\alpha_1)+1;\\ \tfrac{3}{2}, \rho_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{2}, \rho_1-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}, \tfrac{1}{2}(\rho_1+\alpha_1+1), \tfrac{1}{2}(\rho_1+\alpha_1)\end{matrix}\right]\right\}, \qquad (13)
$$

the series ${}_6F_5$ being well poised and terminating when $\rho_1-\alpha_1$ is a negative integer.

Making $\alpha_1\to\infty$ in (13) we get

$$
{}_4F_3\left[\begin{matrix}\alpha_2,\ \alpha_3,\ \rho_1-\tfrac{1}{2};\\ \tfrac{1}{2}(\alpha_2+\alpha_3+1),\ \rho_1\end{matrix}\right]
$$

$$
=\frac{\Gamma[\frac{1}{2}(\alpha_2+\alpha_3+1)]\Gamma(\rho_1)\Gamma(\rho_1-\frac{1}{2}\alpha_2-\frac{1}{2}\alpha_3).\ 2^{\alpha_2+\alpha_3-2}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}
$$

$$
\times\left\{\frac{\Gamma(\frac{1}{2}\alpha_2)\Gamma(\frac{1}{2}\alpha_3)}{\Gamma(\frac{1}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_2)\Gamma(\rho_1-\frac{1}{2}\alpha_3)}\right.
$$

$$
\times\ {}_4F_3\left[\begin{matrix}\rho_1-1,\ \rho_2-\tfrac{1}{2},\ \tfrac{1}{2}\alpha_2,\ \tfrac{1}{2}\alpha_3;\\ \tfrac{1}{2},\ \rho_1-\tfrac{1}{2}\alpha_2,\ \rho_1-\tfrac{1}{2}\alpha_3\end{matrix}\right]
$$

$$
+\frac{(\rho_1-1)\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_2+\frac{1}{2})\Gamma(\rho_1-\frac{1}{2}\alpha_3+\frac{1}{2})}
$$

$$
\left.{}_4F_3\left[\begin{matrix}\rho_1,\ \rho_1-\tfrac{1}{2},\ \tfrac{1}{2}\alpha_2+\tfrac{1}{2},\ \tfrac{1}{2}\alpha_3+\tfrac{1}{2};\\ \tfrac{3}{2},\ \rho_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{2},\ \rho_1-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\end{matrix}\right]\right\}. \quad (14)
$$

3. In Gauss's theorem (3), putting $\beta=-s$, $\gamma=k+s$, $\frac{1}{2}(k-\alpha)=\alpha_1$, $\frac{1}{2}k=\rho_1$ we get

$$
{}_pf_q\left(\begin{matrix}\alpha_1,\ \alpha_1+\tfrac{1}{2},\ \alpha_3,\ \ldots\ldots,\ \alpha_p;\ x\\ \rho_1,\ \rho_1+\tfrac{1}{2},\ 2\alpha_1\ \rho_4,\ \ldots,\ \rho_q\end{matrix}\right)
$$

$$
=\frac{2^{(\rho_1-\alpha_1)}}{\Gamma(2\rho_1-2\alpha_1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-2\alpha_1+r)(-x)^r}{\Gamma(r+1)}
$$

$$
\times\ {}_{p-2}f_{q-2}\left(\begin{matrix}\alpha_3+r,\ \ldots\ldots,\ \alpha_p+r;\ x\\ 2\rho_1+2r,\ \rho_4+r,\ \ldots,\ \rho_q+r\end{matrix}\right), \quad (15)
$$

where $p<q+1$ for all x and $p=q+1$ for $|x|\leqslant 1$.

Taking $x=1$, $p=q=5$, $\rho_2=\alpha_3$ and $\rho_4=\frac{1}{2}(\alpha_3+\alpha_4+1)$ and summing by Watson's theorem (3), we get

$$
{}_4F_3\left[\begin{matrix}\alpha_1,\ \alpha_1+\tfrac{1}{2},\ \alpha_3,\ \alpha_4;\\ 2\alpha_1,\ \rho_1+\tfrac{1}{2},\ \tfrac{1}{2}(\alpha_3+\alpha_4+1)\end{matrix}\right]
$$

$$
=\frac{\Gamma(\rho_1+\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_3+\alpha_4+1)]\Gamma(\frac{1}{2}+\rho_1-\frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4).\ 2^{\alpha_3+\alpha_4-2}}{\Gamma(\frac{1}{2})\Gamma(\alpha_3)\Gamma(\alpha_4)}
$$

$$
\times\left\{\frac{\Gamma(\frac{1}{2}\alpha_3)\Gamma(\frac{1}{2}\alpha_4)}{\Gamma(\frac{1}{2}+\rho_1-\frac{1}{2}\alpha_3)\Gamma(\frac{1}{2}+\rho_1-\frac{1}{2}\alpha_4)}\right.
$$

$$
{}_4F_3\left[\begin{matrix}\rho_1-\alpha_1,\ \rho_1-\alpha_1+\tfrac{1}{2},\ \tfrac{1}{2}\alpha_3,\ \tfrac{1}{2}\alpha_4;\\ \tfrac{1}{2},\ \rho-\tfrac{1}{2}\alpha_3+\tfrac{1}{2},\ \rho_1-\tfrac{1}{2}\alpha_4+\tfrac{1}{2}\end{matrix}\right]
$$

$$- \frac{2(\rho_1-\alpha_1)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_3)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_4)}{\Gamma(1+\rho_1-\frac{1}{2}\alpha_3)\Gamma(1+\rho_1-\frac{1}{2}\alpha_4)}$$

$$\times\ {}_4F_3\left[\begin{matrix}\rho_1-\alpha_1+1,\ \rho_1-\alpha_1+\frac{1}{2},\ \frac{1}{2}\alpha_3+\frac{1}{2},\ \frac{1}{2}\alpha_4+\frac{1}{2};\\ \frac{3}{2},\ \rho_1-\frac{1}{2}\alpha_3+1,\ \rho_1-\frac{1}{2}\alpha_4+1\end{matrix}\right]\Bigg\}, \quad (16)$$

the two series ${}_4F_3$ being "half-poised."

As a particular case, we get on putting $\alpha_1=\rho_1+\frac{1}{2}$

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\alpha_1,\ \alpha_3,\ \alpha_4;\\ 2\alpha_1-1,\ \frac{1}{2}(\alpha_3+\alpha_4+1)\end{matrix}\right]\\ &=\frac{\Gamma(\alpha_1-\frac{1}{2})\Gamma[\frac{1}{2}(\alpha_3+\alpha_4+1)]\Gamma(\alpha_1-\frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4-\frac{1}{2}).\ 2^{\alpha_3+\alpha_4-2}}{\Gamma(\frac{1}{2})\Gamma(\alpha_3)\Gamma(\alpha_4)}\\ &\times\left\{\frac{\Gamma(\frac{1}{2}\alpha_3)\Gamma(\frac{1}{2}\alpha_4)}{\Gamma(\alpha_1-\frac{1}{2}\alpha_3-\frac{1}{2})\Gamma(\alpha_1-\frac{1}{2}\alpha_4-\frac{1}{2})}+\frac{\Gamma(\frac{1}{2}\alpha_3+\frac{1}{2})\Gamma(\frac{1}{2}\alpha_4+\frac{1}{2})}{\Gamma(\alpha_1-\frac{1}{2}\alpha_3)\Gamma(\alpha_1-\frac{1}{2}\alpha_4)}\right\}.\end{aligned} \quad (17)$$

My thanks are due to Dr. S. C. Mitra for his guidance and help in the preparation of the paper.

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